

# Off-shell superconformal nonlinear sigma-models in three dimensions

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## Abstract

We develop superspace techniques to construct general off-shell  $\mathcal{N} \leq 4$  superconformal sigma-models in three space-time dimensions. The most general  $\mathcal{N} = 3$  and  $\mathcal{N} = 4$  superconformal sigma-models are constructed in terms of  $\mathcal{N} = 2$  chiral superfields. Several superspace proofs of the folklore statement that  $\mathcal{N} = 3$  supersymmetry implies  $\mathcal{N} = 4$  are presented both in the on-shell and off-shell settings. We also elaborate on (super)twistor realisations for (super)manifolds on which the three-dimensional  $\mathcal{N}$ -extended superconformal groups act transitively and which include Minkowski space as a subspace.

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# 1 Introduction

Over the last few years the study of superconformal field theories in three space-time dimensions has received a renewed interest. The enthusiasm for this topic has been triggered by the works of Bagger-Lambert [1] and Gustavsson [2] who formulated for the first time a maximally supersymmetric  $\mathcal{N} = 8$  Chern-Simons theory. The analysis undertaken in [1, 2] was aimed at developing a description of the world-volume theory of multiple M2-branes. The same quest has later led to the  $\mathcal{N} = 6$ ,  $U(N) \times U(N)$  superconformal Chern-Simons theory introduced by Aharony, Bergman, Jafferis and Maldacena (ABJM) [3], which is conjectured to describe the low-energy dynamics of a system of  $N$  M2-branes. In the context of the AdS/CFT duality, the ABJM theory in the large- $N$  limit should be dual to the dynamics of M-theory on  $AdS_4 \times S^7/\mathbb{Z}_k$ . At present, extensions of the conjectured duality are a popular subject of investigation in the literature. On the CFT<sub>3</sub> side it would be highly desirable to have a formalism in which a maximally possible amount of supersymmetry is realised off-shell and the superconformal transformations of the multiplets originate in a simple geometric framework. Such properties are useful for investigating various dynamical aspects, including quantum computations, and may be helpful for constructing new nontrivial superconformal field theories. Keeping this in mind, in the present paper we develop superspace formulations for general  $\mathcal{N} \leq 4$  superconformal nonlinear sigma-models in three dimensions. Our main results concern the extended cases  $\mathcal{N} = 3$  and  $\mathcal{N} = 4$ .

Rigid superconformal sigma-models in three dimensions have recently been constructed in component formalism [4] building on earlier works [5, 6, 7, 8, 9, 10]. It was shown, in particular, that for  $\mathcal{N} \leq 4$  the sigma-model target space  $\mathcal{M}$  is a cone with an appropriate Sasakian base manifold. Specifically,  $\mathcal{M}$  is a Riemannian cone for  $\mathcal{N} = 1$  [7], a Kählerian cone for  $\mathcal{N} = 2$  [8], a hyperkähler cone for  $\mathcal{N} = 3$  [9, 10]. If  $\mathcal{N} = 4$ , then  $\mathcal{M}$  factorises into a product of two hyperkähler cones, one parametrised by hypermultiplets and the other by twisted hypermultiplets [4, 6]. In the cases  $\mathcal{N} > 4$ , the superconformal sigma-models were shown in [4] to have necessarily flat target spaces (symmetric target spaces in the locally supersymmetric case [6]). Although the approach of [4] is geometric and insightful, it is intrinsically on-shell. First of all, the superconformal transformations form a closed algebra only on the mass shell. Secondly, and most importantly, this formalism does not allow one to generate superconformal sigma-models with  $\mathcal{N} = 3$  (which in fact implies  $\mathcal{N} = 4$ ). Given a hyperkähler cone [8, 9], one can immediately write down the associated nonlinear sigma-model using the results of [4], but not vice versa. The approach of [4] does not offer sigma-model techniques to generate hyperkähler cones. Such techniques

will be developed in the present paper using the power of superspace model building.

Our approach to construct off-shell superconformal sigma-models in three dimensions is an extension of the 5D  $\mathcal{N} = 1$  and 4D  $\mathcal{N} = 2$  superconformal projective multiplets and their self-couplings [11, 12, 13, 14]. Such multiplets are defined most naturally in projective superspace [15, 16, 17]. The 4D  $\mathcal{N}=2$  projective superspace approach is closely related to the harmonic one [18, 19], for both of them make use of the isotwistor superspace  $\mathbb{R}^{4|8} \times \mathbb{CP}^1 = \mathbb{R}^{4|8} \times S^2$  introduced for the first time by Rosly [20]. The two approaches appear to be complementary in many respects although the latter is more general [21]. As regards sigma-model applications, however, the projective superspace formalism turns out to be more efficient. The point is that sigma-models in harmonic superspace (see [19] for a review) do not possess a natural decomposition in terms of standard 4D  $\mathcal{N} = 1$  superfields, a property that is absolutely essential for various applications. The existence of such a decomposition is one of the powerful inborn features of the 4D  $\mathcal{N} = 2$  multiplets in projective superspace, both in the non-superconformal [16, 17, 22] and superconformal [11, 12, 13, 14, 23] cases. Similar conclusions apply in three dimensions.

It should be pointed out that hyperkähler cones, which are target spaces for 4D  $\mathcal{N} = 2$  [24, 25, 9] and 3D  $\mathcal{N} = 3$  superconformal sigma-models, are intimately related to quaternion Kähler manifolds which are target spaces for 4D  $\mathcal{N} = 2$  [26] and 3D  $\mathcal{N} = 3$  [6] locally supersymmetric  $\sigma$ -models. Specifically, there exists a one-to-one correspondence [27] (see also [28]) between  $4n$ -dimensional quaternion Kähler manifolds and  $4(n+1)$ -dimensional hyperkähler cones. The superspace techniques presented in this paper provide a formalism for constructing  $\mathcal{N} = 3$  rigid superconformal sigma-models generated by a Lagrangian of reasonably general functional form. Then, for any choice of the Lagrangian, the target space metric must be a hyperkähler cone. As a result, the superspace techniques allow us in principle to generate new quaternion Kähler metrics which in general are difficult to construct.

The 3D  $\mathcal{N} = 3$ , 4 superconformal sigma-models, which will be constructed in this paper, can naturally be coupled to conformal supergravity [29], as a generalisation of the approaches developed for 5D  $\mathcal{N} = 1$  [30] and 4D  $\mathcal{N} = 2$  [31] supergravity-matter systems.

The literature on 3D supersymmetric field theories is vast and, unfortunately, we are unable to comment upon many interesting developments. However, we wish to mention a few classic works on off-shell superspace formulations. A thorough study of  $\mathcal{N} = 1$  supersymmetric theories is contained in *Superspace* [32]. Important results on  $\mathcal{N} = 2$  supersymmetric theories appeared in [33]. The  $\mathcal{N} = 3$  harmonic superspace approach

was developed in [34] (it was further reformulated and extended to  $\mathcal{N} = 4$  in [35]).<sup>1</sup> The  $\mathcal{N} = 4$  projective superspace formalism was developed in [33, 16, 17]. We should also mention early papers on 3D  $\mathcal{N} \leq 4$  Chern-Simons gauge theories in superspace [37, 34].

This paper is organised as follows. We start by describing (super)twistor realisations for (super)manifolds on which the three-dimensional  $\mathcal{N}$ -extended superconformal groups act transitively and which include Minkowski space as a subspace.<sup>2</sup> In section 2 we consider the non-supersymmetric case  $\mathcal{N} = 0$  and discuss in detail the structure of compactified Minkowski space. In section 3 we introduce compactified  $\mathcal{N}$ -extended Minkowski superspace. Its harmonic/projective generalisations are presented in section 4. Section 5 is devoted to a thorough analysis of  $\mathcal{N}$ -extended superconformal Killing vectors, and the special cases  $\mathcal{N} \leq 4$  are discussed in much detail. In section 6 we construct general off-shell  $\mathcal{N} = 1$  and  $\mathcal{N} = 2$  superconformal sigma-models. In sections 7 and 8 we derive off-shell  $\mathcal{N} = 3$  and  $\mathcal{N} = 4$  superconformal sigma-models formulated in terms of weight-one polar supermultiplets in projective superspace. Section 9 provides several proofs of the statement that  $\mathcal{N} = 3$  supersymmetry implies  $\mathcal{N} = 4$ . In section 10 we construct the most general  $\mathcal{N} = 3$  and  $\mathcal{N} = 4$  superconformal sigma-models realised in terms of  $\mathcal{N} = 2$  chiral superfields. Concluding comments and discussion are given in section 11. We have also included three technical appendices. Our 3D notation and conventions are collated in Appendix A. Appendix B provides a supermatrix realisation for the  $\mathcal{N}$ -extended super-Poincaré group and Minkowski superspace. Finally, Appendix C describes the  $\mathcal{N} = 2$  reduction for  $\mathcal{N} = 3$  and  $\mathcal{N} = 4$  superconformal Killing vectors.

## 2 Compactified Minkowski space

As is known, the conformal group in  $d$  dimensions does not act globally on Minkowski space  $\mathbb{M}^d \equiv \mathbb{R}^{d-1,1}$ . However, its action is well-defined on a compactified version of Minkowski space  $\overline{\mathbb{M}}^d := (S^{d-1} \times S^1)/\mathbb{Z}_2$ . In this section we present a twistor realisation for  $\overline{\mathbb{M}}^3$  which can be naturally generalised to superspace. The realisation given below can be compared with the twistor construction of  $\overline{\mathbb{M}}^4$  [39, 40, 41, 42, 43] (see [11] for a recent review).

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<sup>1</sup>An interesting application of the 3D  $\mathcal{N} = 3$  harmonic superspace is the recent formulation of the ABJM theory given in [36].

<sup>2</sup>Such (super)manifolds have been discussed by Howe and Leeming [38] in a purely algebraic setting. Our approach elaborates on quite interesting geometric aspects that did not appear in [38].

## 2.1 Twistor construction

Consider a symplectic four-dimensional real vector space. It can be identified with  $\mathbb{R}^4$  equipped with a skew-symmetric inner product:

$$\langle \Psi | \Phi \rangle_J := \Psi^T J \Phi \equiv \Psi_{\hat{\alpha}} J^{\hat{\alpha}\hat{\beta}} \Phi_{\hat{\beta}} = -\langle \Phi | \Psi \rangle_J, \quad J = (J^{\hat{\alpha}\hat{\beta}}) = \begin{pmatrix} 0 & \mathbb{1}_2 \\ -\mathbb{1}_2 & 0 \end{pmatrix}, \quad (2.1)$$

for any vectors  $\Psi, \Phi \in \mathbb{R}^4$ . By construction, this inner product is invariant under the group  $\mathbf{Sp}(2, \mathbb{R})$ . The vectors  $\Psi, \Phi \in \mathbb{R}^4$  will be called *twistors*.<sup>3</sup>

The elements of the group<sup>4</sup>  $\mathbf{Sp}(2, \mathbb{R})$  will be represented by  $4 \times 4$  block matrices

$$g = (g_{\hat{\alpha}}^{\hat{\beta}}) = \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix} \in \mathbf{SL}(4, \mathbb{R}), \quad g^T J g = J, \quad (2.2)$$

where  $\mathcal{A}, \mathcal{B}, \mathcal{C}$  and  $\mathcal{D}$  are  $2 \times 2$  matrices. The symplectic group  $\mathbf{Sp}(2, \mathbb{R})$  is the 2–1 covering of the conformal group in three dimensions,  $\mathbf{SO}_0(3, 2)$ . The group  $\mathbf{Sp}(2, \mathbb{R})$  is generated by its elements of the following three types:

$$\text{Type 1 :} \quad h(\mathcal{A}) := \begin{pmatrix} \mathcal{A} & 0 \\ 0 & (\mathcal{A}^{-1})^T \end{pmatrix}, \quad \mathcal{A} \in \mathbf{GL}(2, \mathbb{R}); \quad (2.3a)$$

$$\text{Type 2 :} \quad s(\mathcal{C}) := \begin{pmatrix} \mathbb{1}_2 & 0 \\ \mathcal{C} & \mathbb{1}_2 \end{pmatrix}, \quad \mathcal{C}^T = \mathcal{C} \in \mathbf{Mat}(2, \mathbb{R}); \quad (2.3b)$$

$$\text{Type 3 :} \quad J. \quad (2.3c)$$

The proof of this result is left to the reader as an instructive exercise.

A *Lagrangian subspace* is defined to be a maximal isotropic vector subspace of  $\mathbb{R}^4$ . Such a subspace is necessarily two-dimensional. We denote by  $\overline{\mathbb{M}}^3$  the space of all Lagrangian subspaces of  $\mathbb{R}^4$ . The group  $\mathbf{Sp}(2, \mathbb{R})$  proves to act transitively on the compact space  $\overline{\mathbb{M}}^3$  which can be realised as

$$\overline{\mathbb{M}}^3 = \mathbf{U}(2)/\mathbf{O}(2), \quad (2.4)$$

see, e.g., [44] for technical details. This homogeneous space with the transformation group  $\mathbf{Sp}(2, \mathbb{R})$  is called the *compactified Minkowski space*.

<sup>3</sup>In four space-time dimensions, twistors are necessarily complex, see e.g. [40, 43].

<sup>4</sup>The group  $\mathbf{Sp}(2, \mathbb{R})$  should not be confused with its compact sister  $\mathbf{Sp}(2) := \mathbf{Sp}(2, \mathbb{C}) \cap \mathbf{U}(4)$ . Sometimes, the group  $\mathbf{Sp}(2, \mathbb{R})$  is denoted  $\mathbf{Sp}(4, \mathbb{R})$ , and similarly for its complexified version.

Let  $\mathcal{L} \in \overline{\mathbb{M}}^3$  be a Lagrangian subspace. It is generated by two linearly independent twistors  $T^\mu$ , with  $\mu = 1, 2$ , such that

$$\langle T^1 | T^2 \rangle_J = 0 . \quad (2.5)$$

Obviously, the basis chosen,  $\{T^\mu\}$ , is defined only modulo the equivalence relation

$$\{T^\mu\} \sim \{\tilde{T}^\mu\} , \quad \tilde{T}^\mu = T^\nu R_\nu{}^\mu , \quad R \in \text{GL}(2, \mathbb{R}) . \quad (2.6)$$

Equivalently, we can think of the space  $\overline{\mathbb{M}}^3$  as consisting of rank-two  $4 \times 2$  real matrices

$$(T^1 \ T^2) = \begin{pmatrix} F \\ G \end{pmatrix} , \quad F^T G = G^T F , \quad (2.7)$$

where the  $2 \times 2$  matrices  $F$  and  $G$  are defined modulo the equivalence relation

$$\begin{pmatrix} F \\ G \end{pmatrix} \sim \begin{pmatrix} F R \\ G R \end{pmatrix} , \quad R \in \text{GL}(2, \mathbb{R}) . \quad (2.8)$$

An open dense subset  $\mathbb{M}^3$  of  $\overline{\mathbb{M}}^3$  consists of those Lagrangian subspaces which are described by  $4 \times 2$  matrices of the form:

$$\begin{pmatrix} F \\ G \end{pmatrix} , \quad \det F \neq 0 . \quad (2.9)$$

In accordance with the equivalence relation (2.8), we then have

$$\begin{pmatrix} F \\ G \end{pmatrix} \sim \begin{pmatrix} \mathbb{1}_2 \\ -x \end{pmatrix} , \quad x^T = x \in \text{Mat}(2, \mathbb{R}) . \quad (2.10)$$

The subset  $\mathbb{M}^3$  can naturally be identified with Minkowski space  $\mathbb{R}^{2,1}$ , as demonstrated in Appendix B. In what follows, we will not distinguish between  $\mathbb{M}^3$  and  $\mathbb{R}^{2,1}$ .

## 2.2 The conformal infinity of Minkowski space

Let us analyse the structure of the boundary of Minkowski space in  $\overline{\mathbb{M}}^3$ , that is

$$\partial \mathbb{M}^3 = \overline{\mathbb{M}}^3 \setminus \mathbb{M}^3 . \quad (2.11)$$

For any point from  $\partial \mathbb{M}^3$ , the  $2 \times 2$  block  $F$  is singular,  $\det F = 0$ . Because of the equivalence relation (2.8), we can always choose

$$F = \begin{pmatrix} f_1 & 0 \\ f_2 & 0 \end{pmatrix} , \quad (2.12)$$



for some two-vector  $\vec{f} \in \mathbb{R}^2$ . There are two different cases to consider: (i)  $\vec{f} = 0$ ; and (ii)  $\vec{f} \neq 0$ . In the case that  $\vec{f} = 0$ , we have  $\det G \neq 0$ , and thus for the corresponding Lagrangian subspace

$$\begin{pmatrix} 0 \\ G \end{pmatrix} \sim \begin{pmatrix} 0 \\ \mathbb{1}_2 \end{pmatrix} . \quad (2.13)$$

As a result, the case (i) leads to a single point in  $\partial \mathbb{M}^3$ . The case (ii) is more interesting. Making use of the equivalence relation (2.8), we can normalise  $\vec{f}$  by

$$\vec{f} \cdot \vec{f} = 1 . \quad (2.14)$$

We still can change the sign of the vector,  $\vec{f} \rightarrow -\vec{f}$ , which amounts to residual  $\mathbb{Z}_2$ -freedom in the choice of  $\vec{f}$ . For the matrix

$$G = \begin{pmatrix} u_1 & g_1 \\ u_2 & g_2 \end{pmatrix} , \quad \vec{g} \neq 0 , \quad (2.15)$$

the isotropy condition (2.7) gives

$$\vec{f} \cdot \vec{g} = 0 . \quad (2.16)$$

Making use of the equivalence relation (2.8), we can impose the normalisation condition

$$\vec{g} \cdot \vec{g} = 1 . \quad (2.17)$$

Moreover, the same equivalence relation still allows us to choose the direction of  $\vec{g}$  such that the set  $\{\vec{f}, \vec{g}\}$  is an orthonormal basis of standard orientation in  $\mathbb{R}^2$ . In other words, the vector  $\vec{g}$  is uniquely determined by the choice of  $\vec{f}$ . With the vectors  $\{\vec{f}, \vec{g}\}$  having been fixed, the freedom (2.8) still allows to perform transformations

$$\vec{u} \rightarrow \vec{u} + c \vec{g} , \quad c \in \mathbb{R} . \quad (2.18)$$

Thus, we can make  $\vec{u}$  to be a multiple of  $\vec{f}$ . We conclude that any Lagrangian subspace from the boundary (2.11) looks like

$$\begin{pmatrix} F \\ G \end{pmatrix} \sim \begin{pmatrix} f_1 & 0 \\ f_2 & 0 \\ \lambda f_1 & g_1 \\ \lambda f_2 & g_2 \end{pmatrix} , \quad \lambda \in \mathbb{R} , \quad (2.19)$$

where the two-vectors  $\vec{f}$  and  $\vec{g}$  form a standard orthonormal frame in  $\mathbb{R}^2$ . The space of two-frames  $\{\vec{f}, \vec{g}\}$  is topologically  $S^1$ . On the other hand, in the limit  $\lambda \rightarrow \pm\infty$  the Lagrangian subspace (2.19) can be seen to turn into that defined in the case (i) above. as a result, we see that the boundary (2.11) is topologically

$$\overline{\mathbb{M}}^3 \setminus \mathbb{M}^3 = (S^1 \times S^1)/\mathbb{Z}_2 . \quad (2.20)$$

## 2.3 The conformal algebra

This subsection describes a useful matrix realisation of the conformal algebra  $\mathfrak{sp}(2, \mathbb{R}) \cong \mathfrak{so}(3, 2)$ . In what follows we will use the following notation:

$$\gamma_a := (\gamma_a)_\alpha{}^\beta, \quad \hat{\gamma}_a := (\gamma_a)^{\alpha\beta} = \hat{\gamma}_a^T, \quad \tilde{\gamma}_a := (\gamma_a)_{\alpha\beta} = \tilde{\gamma}_a^T. \quad (2.21)$$

Defining also  $\varepsilon = (\varepsilon^{\alpha\beta})$ , the gamma-matrices are characterised by the property

$$\gamma_a^T = -\varepsilon \gamma_a \varepsilon^{-1}. \quad (2.22)$$

Introduce  $4 \times 4$  matrices  $\Gamma_A$ , with  $A = a, 3, 4$ , of the form:

$$\Gamma_a = \begin{pmatrix} \gamma_a & 0 \\ 0 & (\gamma_a)^T \end{pmatrix}, \quad \Gamma_3 = \begin{pmatrix} 0 & \varepsilon^{-1} \\ \varepsilon & 0 \end{pmatrix}, \quad \Gamma_4 = \begin{pmatrix} 0 & -\varepsilon^{-1} \\ \varepsilon & 0 \end{pmatrix}. \quad (2.23)$$

They obey the anti-commutation relations

$$\{\Gamma_A, \Gamma_B\} = 2\eta_{AB} \mathbb{1}_4, \quad \eta_{AB} = \text{diag}(- + + + -). \quad (2.24)$$

Clearly, the matrices  $\Gamma_A$  constitute a Majorana representation of the gamma-matrices for pseudo-Euclidean space  $\mathbb{R}^{3,2}$ . One easily checks that

$$\Gamma_A^T = J \Gamma_A J^{-1}, \quad (2.25)$$

with  $J = \Gamma_0 \Gamma_4$  the symplectic matrix (2.1).

As follows from (2.24), the matrices

$$\Sigma_{AB} := \frac{1}{4} [\Gamma_A, \Gamma_B] \quad (2.26)$$

form the spinor representation of  $\mathfrak{so}(3, 2)$ . It also follows from (2.25) that  $\Sigma_{AB}$  obey the relations

$$\Sigma_{AB}^T J + J \Sigma_{AB} = 0, \quad (2.27)$$

and therefore they form a basis of  $\mathfrak{sp}(2, \mathbb{R})$ . That is, for any  $\omega \in \mathfrak{sp}(2, \mathbb{R})$ , such that  $\omega^T J + J \omega = 0$ , we have

$$\omega = \frac{1}{2} \omega^{AB} \Sigma_{AB}, \quad \omega^{AB} = -\omega^{BA} \in \mathbb{R}. \quad (2.28)$$

The above consideration immediately leads to the famous result  $\mathfrak{sp}(2, \mathbb{R}) \cong \mathfrak{so}(3, 2)$ .

Given an arbitrary element  $\omega \in \mathfrak{sp}(2, \mathbb{R})$ , it can be represented as

$$\omega = \left( \begin{array}{c|c} \lambda_\alpha^\beta - \frac{1}{2}f\delta_\alpha^\beta & b_{\alpha\beta} \\ \hline -a^{\alpha\beta} & -\lambda^\alpha_\beta + \frac{1}{2}f\delta^\alpha_\beta \end{array} \right) \equiv \left( \begin{array}{c|c} \lambda - \frac{1}{2}f\mathbb{1}_2 & \check{b} \\ \hline -\hat{a} & -\lambda^T + \frac{1}{2}f\mathbb{1}_2 \end{array} \right) \in \mathfrak{sp}(2, \mathbb{R}) ,$$

$$\lambda_\alpha^\alpha = 0 , \quad a^{\alpha\beta} = a^{\beta\alpha} , \quad b_{\alpha\beta} = b_{\beta\alpha} . \quad (2.29)$$

Associated with  $J$  is the following automorphism of the conformal algebra:

$$J \left( \begin{array}{c|c} \lambda - \frac{1}{2}f\mathbb{1}_2 & \check{b} \\ \hline -\hat{a} & -\lambda^T + \frac{1}{2}f\mathbb{1}_2 \end{array} \right) J^{-1} = \left( \begin{array}{c|c} -\lambda^T + \frac{1}{2}f\mathbb{1}_2 & \hat{a} \\ \hline -\check{b} & \lambda - \frac{1}{2}f\mathbb{1}_2 \end{array} \right) . \quad (2.30)$$

## 2.4 Compactified Minkowski space à la Dirac

Let  $\mathcal{L}$  be a Lagrangian subspace of  $\mathbb{R}^4$  generated by two linearly independent twistors  $T^1$  and  $T^2$ . It follows from (2.25) that

$$\langle T^\mu | \Gamma_A | T^\nu \rangle_J = -\langle T^\nu | \Gamma_A | T^\mu \rangle_J . \quad (2.31)$$

Applying the equivalence transformation (2.6) to  $\langle T^\mu | \Gamma_A | T^\nu \rangle_J$  gives

$$\langle \tilde{T}^\mu | \Gamma_A | \tilde{T}^\nu \rangle_J = \det R \langle T^\mu | \Gamma_A | T^\nu \rangle_J . \quad (2.32)$$

As follows from the identity

$$(J\Gamma^A)^{\hat{\alpha}\hat{\beta}}(J\Gamma_A)^{\hat{\gamma}\hat{\delta}} = -J^{\hat{\alpha}\hat{\beta}}J^{\hat{\gamma}\hat{\delta}} + 2(J^{\hat{\alpha}\hat{\gamma}}J^{\hat{\beta}\hat{\delta}} - J^{\hat{\alpha}\hat{\delta}}J^{\hat{\beta}\hat{\gamma}}) , \quad (2.33)$$

the five-vector

$$X^A := \langle T^1 | \Gamma^A | T^2 \rangle_J \quad (2.34)$$

belongs to the cone  $\mathcal{C}$  in  $\mathbb{R}^{3,2}$  defined by

$$\eta_{AB}X^AX^B = 0 . \quad (2.35)$$

It is an instructive exercise to demonstrate that any null vector  $X^A \in \mathcal{C}$  can be represented in the form (2.34), for some null twistors  $T^1$  and  $T^2$  which generate a Lagrangian subspace of  $\mathbb{R}^4$ .

It follows from the above consideration that the five-vector (2.34) is defined modulo the equivalence relation

$$X^A \sim \lambda X^A , \quad \lambda \in \mathbb{R} - \{0\} \quad (2.36)$$

which identifies all points on a straight line in  $\mathbb{R}^{3,2}$ . The space of all straight lines through the origin of the cone  $\mathcal{C}$  in  $\mathbb{R}^{3,2}$ , eq. (2.35), is known as Dirac's conformal space<sup>5</sup> [45] (although it had been introduced a decade earlier by Weyl [46]) which is topologically  $(S^2 \times S^1)/\mathbb{Z}_2$ . We conclude that  $\overline{\mathbb{M}}^3$  can be identified with Dirac's conformal space,

$$\overline{\mathbb{M}}^3 = (S^2 \times S^1)/\mathbb{Z}_2 . \quad (2.37)$$

### 3 Compactified Minkowski superspace

In this section we generalise the construction of 3D compactified Minkowski space to  $\mathcal{N}$ -extended superspace. Our presentation will be similar in spirit to that given in [11] for the case of four space-time dimensions. In its turn, the work of [11] built on Manin's approach to the 4D  $\mathcal{N}$ -extended superconformal symmetry [47] in conjunction with Ferber's concept of supertwistors [48].

#### 3.1 The superconformal group

Associated with the symplectic super-metric

$$\mathbb{J} = \begin{pmatrix} J & 0 \\ 0 & i \mathbb{1}_{\mathcal{N}} \end{pmatrix} \quad (3.1)$$

is the symmetric purely imaginary quadratic form

$$\Sigma^{\text{sT}} \mathbb{J} \Sigma = \zeta^{\text{T}} J \zeta + i y^{\text{T}} y , \quad (3.2)$$

which is defined on the superspace  $\mathbb{R}^{\mathcal{N}|4}$  parametrised by 4 anticommuting real variables  $\zeta$  and  $\mathcal{N}$  commuting real variables  $y$ ,

$$\Sigma = \begin{pmatrix} \zeta \\ y \end{pmatrix} , \quad \Sigma^{\text{sT}} = (\zeta^{\text{T}} , y^{\text{T}}) = \Sigma^{\text{T}} , \quad \epsilon(\zeta) = 1 , \quad \epsilon(y) = 0 . \quad (3.3)$$

Here and in what follows  $\epsilon(s)$  denotes the Grassmann parity of a supernumber  $s$ . Any element  $\Sigma \in \mathbb{R}^{\mathcal{N}|4}$  of the form (3.3) will be called an *odd real supertwistor*. Given a linear transformation

$$z \rightarrow z' = g z , \quad g = \left( \frac{A \parallel B}{C \parallel D} \right) , \quad (3.4)$$

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<sup>5</sup>In [45] Dirac credited Oswald Veblen with the general theory of conformal space.

it leaves the quadratic form (3.2) invariant if the supermatrix  $g$  obeys the equation

$$g^{sT} \mathbb{J} g = \mathbb{J} , \quad g^{sT} = \left( \begin{array}{c|c} A^T & -C^T \\ \hline B^T & D^T \end{array} \right) . \quad (3.5)$$

Such supermatrices form the supergroup  $\text{OSp}(\mathcal{N}|2, \mathbb{R})$  which is the  $\mathcal{N}$ -extended superconformal group in three space-time dimensions. The even matrices  $A, D$  and the odd matrix  $B$  in (3.4) have real matrix elements, while the odd matrix  $C$  has purely imaginary matrix elements. Supermatrices  $g$  of this type will be called *real*.

The quadratic form (3.2) can naturally be extended to the symmetric inner product on  $\mathbb{R}^{\mathcal{N}|4}$  defined by

$$\langle \Sigma | \Xi \rangle_{\mathbb{J}} := \Sigma^{sT} \mathbb{J} \Xi = \langle \Xi | \Sigma \rangle_{\mathbb{J}} , \quad (3.6)$$

with  $\Sigma$  and  $\Xi$  being arbitrary odd supertwistors. This inner product is obviously invariant under the action of  $\text{OSp}(\mathcal{N}|2, \mathbb{R})$ .

The superconformal algebra  $\mathfrak{osp}(\mathcal{N}|2, \mathbb{R})$  consists of real supermatrices obeying the master equation

$$\omega^{sT} \mathbb{J} + \mathbb{J} \omega = 0 . \quad (3.7)$$

The general solution of this equation is

$$\begin{aligned} \omega &= \left( \begin{array}{c|c|c} \lambda - \frac{1}{2} f \mathbb{1}_2 & \check{b} & \sqrt{2} \eta^T \\ \hline -\hat{a} & -\lambda^T + \frac{1}{2} f \mathbb{1}_2 & -\sqrt{2} \epsilon^T \\ \hline i\sqrt{2} \epsilon & i\sqrt{2} \eta & r \end{array} \right) \\ &\equiv \left( \begin{array}{c|c|c} \lambda_{\alpha}^{\beta} - \frac{1}{2} f \delta_{\alpha}^{\beta} & b_{\alpha\beta} & \sqrt{2} \eta_{\alpha J} \\ \hline -a^{\alpha\beta} & -\lambda^{\alpha}_{\beta} + \frac{1}{2} f \delta^{\alpha}_{\beta} & -\sqrt{2} \epsilon^{\alpha J} \\ \hline i\sqrt{2} \epsilon_I^{\beta} & i\sqrt{2} \eta_{I\beta} & r_{IJ} \end{array} \right) , \quad I, J = 1, \dots, \mathcal{N} \quad (3.8) \\ \lambda_{\alpha}^{\alpha} &= 0 , \quad a^{\alpha\beta} = a^{\beta\alpha} , \quad b_{\alpha\beta} = b_{\beta\alpha} , \quad r_{IJ} = -r_{JI} . \end{aligned}$$

The bosonic parameters  $\lambda_{\alpha}^{\beta}$ ,  $f$ ,  $a_{\alpha\beta}$ ,  $b^{\alpha\beta}$  and  $r_{IJ}$ , as well as the fermionic parameters  $\epsilon^{\alpha}_I \equiv \epsilon_I^{\alpha}$  and  $\eta_{\alpha I} \equiv \eta_{I\alpha}$  in (3.8) are real.

### 3.2 Compactified Minkowski superspace

The superconformal group  $\text{OSp}(\mathcal{N}|2, \mathbb{R})$  naturally acts on  $\mathbb{R}^{4|\mathcal{N}}$  parametrised by elements of the form:

$$\mathbf{S} = \begin{pmatrix} f_\alpha \\ g^\alpha \\ \mathbf{i} \varphi_I \end{pmatrix}, \quad \epsilon(f_\alpha) = \epsilon(g^\alpha) = 0, \quad \epsilon(\varphi_I) = 1, \quad (3.9)$$

with the components  $f_\alpha$  and  $g^\alpha$  being real commuting, and  $\varphi_I$  real anticommuting. This action preserves the inner product on  $\mathbb{R}^{4|\mathcal{N}}$  defined by

$$\langle \mathbf{S} | \mathbf{T} \rangle_{\mathbb{J}} := \mathbf{S}^{\text{sT}} \mathbb{J} \mathbf{T} = -\langle \mathbf{T} | \mathbf{S} \rangle_{\mathbb{J}}, \quad \mathbf{S}^{\text{sT}} = (f_\alpha, g^\alpha, -\mathbf{i} \varphi_I), \quad (3.10)$$

with the super-metric  $\mathbb{J}$  defined in (3.1). Any element  $\mathbf{S} \in \mathbb{R}^{4|\mathcal{N}}$  is called an *even real supertwistor*.

By analogy with the non-supersymmetric case, we define a *Lagrangian subspace* of  $\mathbb{R}^{4|\mathcal{N}}$  to be its maximal isotropic subspace. We denote by  $\overline{\mathbb{M}}^{3|2\mathcal{N}}$  the space of all Lagrangian subspaces of  $\mathbb{R}^{4|\mathcal{N}}$ . Given such a subspace, it is generated by two supertwistors  $\mathbf{T}^\mu$  such that (i) the bodies of  $\mathbf{T}^1$  and  $\mathbf{T}^2$  are linearly independent; (ii) they obey the null condition

$$\langle \mathbf{T}^1 | \mathbf{T}^2 \rangle_{\mathbb{J}} = 0; \quad (3.11)$$

(iii) they are defined only modulo the equivalence relation

$$\{\mathbf{T}^\mu\} \sim \{\tilde{\mathbf{T}}^\mu\}, \quad \tilde{\mathbf{T}}^\mu = \mathbf{T}^\nu R_\nu{}^\mu, \quad R \in \text{GL}(2, \mathbb{R}). \quad (3.12)$$

Equivalently, the space  $\overline{\mathbb{M}}^{3|2\mathcal{N}}$  consists of rank-two supermatrices of the form

$$(\mathbf{T}^1 \ \mathbf{T}^2) = \begin{pmatrix} F \\ G \\ \mathbf{i} \Upsilon \end{pmatrix}, \quad G^{\text{T}} F = F^{\text{T}} G + \mathbf{i} \Upsilon^{\text{T}} \Upsilon, \quad (3.13)$$

which are defined modulo the equivalence relation

$$\begin{pmatrix} F \\ G \\ \mathbf{i} \Upsilon \end{pmatrix} \sim \begin{pmatrix} F R \\ G R \\ \mathbf{i} \Upsilon R \end{pmatrix}, \quad R \in \text{GL}(2, \mathbb{R}). \quad (3.14)$$

Here  $F$  and  $G$  are  $2 \times 2$  real bosonic matrices, and  $\Upsilon$  is a  $\mathcal{N} \times 2$  real fermionic matrix.

A dense open subset  $\mathbb{M}^{3|2\mathcal{N}}$  of  $\overline{\mathbb{M}}^{3|2\mathcal{N}}$  consists of those Lagrangian subspaces which are described by supermatrices (3.13) under the condition

$$\det F \neq 0 . \quad (3.15)$$

Using the equivalence relation (3.14), such a supermatrix can be brought to the following canonical form:

$$\begin{pmatrix} \mathbf{T}^1 & \mathbf{T}^2 \end{pmatrix} = \begin{pmatrix} F \\ G \\ \mathbf{i} \Upsilon \end{pmatrix} \sim \begin{pmatrix} \delta_\alpha^\beta \\ -x^{\alpha\beta} + \frac{\mathbf{i}}{2} \varepsilon^{\alpha\beta} \theta^2 \\ \mathbf{i} \sqrt{2} \theta_I^\beta \end{pmatrix} , \quad x^{\alpha\beta} = x^{\beta\alpha} , \quad \theta^2 := \theta_I^\alpha \theta_{\alpha I} . \quad (3.16)$$

Here the bosonic  $x^{\alpha\beta}$  and fermionic  $\theta_I^\alpha \equiv \theta_I^\alpha$  parameters are real. Therefore, the subset  $\mathbb{M}^{3|2\mathcal{N}} \subset \overline{\mathbb{M}}^{3|2\mathcal{N}}$  defined by eq. (3.15) can be identified with  $\mathbb{R}^{3|2\mathcal{N}}$ .

Given a group element  $g \in \text{OSp}(\mathcal{N}|2, \mathbb{R})$ , its (local) action on the points of  $\mathbb{M}^{3|2\mathcal{N}}$  can be derived by the rule

$$g \begin{pmatrix} \mathbb{1}_2 \\ -x_{(-)} \\ \mathbf{i} \sqrt{2} \theta \end{pmatrix} = \begin{pmatrix} \mathbb{1}_2 \\ -x'_{(-)} \\ \mathbf{i} \sqrt{2} \theta' \end{pmatrix} R(g; x, \theta) , \quad R(g; x, \theta) \in \text{GL}(2, \mathbb{R}) \quad (3.17)$$

where we have denoted

$$x_{(-)} := \hat{x} - \frac{\mathbf{i}}{2} \varepsilon \theta^2 = \left( x^{\alpha\beta} - \frac{\mathbf{i}}{2} \varepsilon^{\alpha\beta} \theta^2 \right) , \quad \theta := (\theta_I^\beta) . \quad (3.18)$$

In the case of an infinitesimal superconformal transformation (3.8), from (3.17) we derive

$$\delta x_{(-)} = \hat{a} - \lambda^T x_{(-)} - x_{(-)} \lambda + f x_{(-)} + x_{(-)} \check{b} x_{(-)} + 2\mathbf{i} \epsilon^T \theta - 2\mathbf{i} x_{(-)} \eta^T \theta , \quad (3.19a)$$

$$\delta \theta = \epsilon - \theta \lambda + \frac{1}{2} f \theta + r \theta + \theta \check{b} x_{(-)} - 2\mathbf{i} \theta \eta^T \theta . \quad (3.19b)$$

These relations can be rewritten as

$$\delta x^m = \xi^m + \mathbf{i} \xi_I^\alpha (\gamma^m)_{\alpha\beta} \theta_I^\beta , \quad \delta \theta_I^\alpha = \xi_I^\alpha , \quad (3.20)$$

where  $\xi^m(x, \theta)$  and  $\xi_I^\alpha(x, \theta)$  are the components of a superconformal Killing vector field. These objects will be studied in section 5.

## 4 Compactified harmonic/projective superspaces

We wish to construct new homogeneous spaces,  $\overline{\mathbb{M}}^{3|2\mathcal{N}} \times \mathbb{X}_m^\mathcal{N}$ , of the superconformal group  $\text{OSp}(\mathcal{N}|2, \mathbb{R})$  that include  $\overline{\mathbb{M}}^{3|2\mathcal{N}}$  as a submanifold, for any integer  $m \leq [\mathcal{N}/2]^6$ .

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<sup>6</sup>As usual, we denote by  $[\mathcal{N}/2]$  the integer part of  $\mathcal{N}/2$ .

Our approach will be analogous to the four-dimensional construction of [11] which built on earlier works [49, 50, 51].

## 4.1 Projective realisation

Along with the two linearly independent even real supertwistors  $\mathbf{T}^1$  and  $\mathbf{T}^2$  under the null condition (3.11), we also consider  $m$  odd *complex* supertwistors  $\Sigma^{\underline{i}}$ , with  $\underline{i} = 1, \dots, m$ , such that (i) the bodies of  $\Sigma^{\underline{i}}$  are linearly independent; (ii) any linear combination of the supertwistors  $\mathbf{T}^\mu$  and  $\Sigma^{\underline{i}}$  is null, that is

$$\langle \mathbf{T}^\mu | \mathbf{T}^\nu \rangle_{\mathbb{J}} = \langle \mathbf{T}^\mu | \Sigma^{\underline{j}} \rangle_{\mathbb{J}} = \langle \Sigma^{\underline{i}} | \Sigma^{\underline{j}} \rangle_{\mathbb{J}} = 0 . \quad (4.1)$$

The supertwistors  $\mathbf{T}^\mu$  and  $\Sigma^{\underline{i}}$  are assumed to be defined modulo the equivalence relation

$$(\mathbf{T}^\mu, \Sigma^{\underline{i}}) \sim (\mathbf{T}^\nu, \Sigma^{\underline{j}}) \left( \begin{array}{c|c} R_\nu^\mu & B_\nu^{\underline{i}} \\ \hline 0 & D_{\underline{j}}^{\underline{i}} \end{array} \right) , \quad \left( \begin{array}{c|c} R & B \\ \hline 0 & D \end{array} \right) \in \text{GL}(2|m, \mathbb{C}) , \quad R \in \text{GL}(2, \mathbb{R}) . \quad (4.2)$$

We emphasise that both the fermionic  $B_\nu^{\underline{i}}$  and bosonic  $D_{\underline{j}}^{\underline{i}}$  matrix elements are complex. The space  $\overline{\mathbb{M}}^{3|2\mathcal{N}} \times \mathbb{X}_m^{\mathcal{N}}$  is defined to consist of the equivalence classes associated with all possible  $(\mathbf{T}^\mu, \Sigma^{\underline{i}})$  under the above conditions.

There are several important comments to be made. Firstly, the invariant inner product  $\langle , \rangle_{\mathbb{J}}$  possesses the following symmetry property

$$\langle \mathcal{T}_1 | \mathcal{T}_2 \rangle_{\mathbb{J}} = -(-1)^{\epsilon_1 \epsilon_2} \langle \mathcal{T}_2 | \mathcal{T}_1 \rangle_{\mathbb{J}} , \quad (4.3)$$

where  $\epsilon_i$  denotes the Grassmann parity of  $\mathcal{T}_i$ . Secondly, associated with the odd supertwistors  $\Sigma^{\underline{i}}$  are their complex conjugates  $\bar{\Sigma}^{\underline{j}}$  which possess analogous properties

$$\langle \mathbf{T}^\mu | \bar{\Sigma}^{\underline{j}} \rangle_{\mathbb{J}} = \langle \bar{\Sigma}^{\underline{i}} | \bar{\Sigma}^{\underline{j}} \rangle_{\mathbb{J}} = 0 . \quad (4.4)$$

One can also see that the  $2m$  supertwistors  $\Sigma^{\underline{i}}$  and  $\bar{\Sigma}^{\underline{j}}$  are linearly independent,

$$\det \langle \Sigma^{\underline{i}} | \bar{\Sigma}^{\underline{j}} \rangle_{\mathbb{J}} \neq 0 . \quad (4.5)$$

We are mostly interested in the dense open subset  $\mathbb{M}^{3|2\mathcal{N}} \times \mathbb{X}_m^{\mathcal{N}}$  of  $\overline{\mathbb{M}}^{3|2\mathcal{N}} \times \mathbb{X}_m^{\mathcal{N}}$ . For its points, the even supertwistors  $\mathbf{T}^\mu$  can be chosen as in (3.16). Making use of the null conditions (4.1) and the equivalence relation (4.2), it is not difficult to show that the points of  $\mathbb{M}^{3|2\mathcal{N}} \times \mathbb{X}_m^{\mathcal{N}}$  can be parametrised by supermatrices of the form:

$$(\mathbf{T}^1 \ \mathbf{T}^2) \sim \begin{pmatrix} \delta_\alpha^\beta & \\ -x^{\alpha\beta} + \frac{i}{2}\varepsilon^{\alpha\beta}\theta^2 & \\ i\sqrt{2}\theta_I^\beta & \end{pmatrix} , \quad (\Sigma^1 \dots \Sigma^m) \sim \begin{pmatrix} 0 & \\ -\sqrt{2}\theta_K^\alpha Z_K^{\underline{i}} & \\ Z_I^{\underline{i}} & \end{pmatrix} . \quad (4.6)$$



Here the  $m$  complex vectors  $Z^{\underline{j}} = (Z_I^{\underline{j}}) \in \mathbb{C}^{\mathcal{N}} - \{0\}$  are required to be linearly independent and subject to the null conditions

$$Z^{\underline{j}} \cdot Z^{\underline{k}} := Z_I^{\underline{j}} Z_I^{\underline{k}} = 0, \quad \forall \underline{j}, \underline{k} = 1, \dots, m \quad (4.7)$$

and are defined modulo the equivalence relation

$$Z_I^{\underline{j}} \sim Z_I^{\underline{k}} D_{\underline{k}}^{\underline{j}}, \quad D = (D_{\underline{k}}^{\underline{j}}) \in \text{GL}(m, \mathbb{C}) . \quad (4.8)$$

The complex  $\mathcal{N}$ -vectors  $Z^{\underline{j}}$  can be represented as a superposition of their real and imaginary parts,  $Z^{\underline{j}} = \frac{1}{\sqrt{2}}(X^{\underline{j}} + iY^{\underline{j}})$ . Then, the null conditions (4.7) take the form:

$$X^{\underline{j}} \cdot X^{\underline{k}} = Y^{\underline{j}} \cdot Y^{\underline{k}}, \quad X^{\underline{j}} \cdot Y^{\underline{k}} = -X^{\underline{k}} \cdot Y^{\underline{j}}, \quad \forall \underline{j}, \underline{k} = 1, \dots, m . \quad (4.9)$$

## 4.2 Harmonic realisation

We would like to describe an alternative realisation of the space  $\overline{\mathbb{M}}^{3|2\mathcal{N}} \times \mathbb{X}_m^{\mathcal{N}}$ . It differs from the projective realisation considered above, but is equivalent to it. Along with the two linearly independent even real supertwistors  $\mathbf{T}^1$  and  $\mathbf{T}^2$  and the  $m$  odd *complex* supertwistors  $\Sigma^{\underline{i}}$  under the null conditions (4.1), we also consider  $\mathcal{N} - m \geq m$  odd *complex* supertwistors  $\Upsilon^{\underline{I}}$ , with  $\underline{I} = 1, \dots, \mathcal{N} - m$ , such that (i) the bodies of  $\Sigma^{\underline{i}}$  and  $\Upsilon^{\underline{I}}$  form a basis for  $\mathbb{C}^{\mathcal{N}}$ ; (ii) the odd supertwistors  $\Upsilon^{\underline{I}}$  are orthogonal to the even supertwistors,

$$\langle \mathbf{T}^{\mu} | \Upsilon^{\underline{J}} \rangle_{\mathbb{J}} = 0 . \quad (4.10)$$

The set of  $2 + \mathcal{N}$  supertwistors  $\mathbf{T}^{\mu}$ ,  $\Sigma^{\underline{i}}$  and  $\Upsilon^{\underline{I}}$  is assumed to be defined modulo the equivalence relation:

$$(\mathbf{T}^{\mu}, \Sigma^{\underline{i}}, \Upsilon^{\underline{I}}) \sim (\mathbf{T}^{\nu}, \Sigma^{\underline{j}}, \Upsilon^{\underline{J}}) \left( \begin{array}{c|cc} R_{\nu}^{\mu} & B_{\nu}^{\underline{i}} & C_{\nu}^{\underline{I}} \\ \hline 0 & D_{\underline{j}}^{\underline{i}} & E_{\underline{j}}^{\underline{I}} \\ \hline 0 & 0 & F_{\underline{J}}^{\underline{I}} \end{array} \right) . \quad (4.11a)$$

Here the  $(2 + \mathcal{N}) \times (2 + \mathcal{N})$  supermatrix on the right is chosen such that

$$\left( \begin{array}{c|cc} R & B & C \\ \hline 0 & D & E \\ \hline 0 & 0 & F \end{array} \right) \in \text{GL}(2|\mathcal{N}, \mathbb{C}) , \quad R \in \text{GL}(2, \mathbb{R}) \quad (4.11b)$$

but otherwise it is arbitrary. It is not difficult to show that the space  $\overline{\mathbb{M}}^{3|2\mathcal{N}} \times \mathbb{X}_m^{\mathcal{N}}$  can be identified with the space of equivalence classes associated with all possible triplets  $(\mathbf{T}^\mu, \Sigma^i, \Upsilon^\perp)$  under the above conditions.

We consider the Minkowski patch and choose the isotwistors  $\mathbf{T}^\mu$  and  $\Sigma^i$  as in (4.6). The equivalence relation (4.11a) allows us to choose the  $\mathcal{N} - m$  odd supertwistors  $\Upsilon^\perp$  in the form:

$$(\Upsilon^1 \dots \Upsilon^{\mathcal{N}-m}) \sim \begin{pmatrix} 0 \\ -\sqrt{2}\theta_K^\alpha W_{K^\perp} \\ W_{I^\perp} \end{pmatrix}, \quad (4.12)$$

for some complex  $\mathcal{N}$ -vectors  $W^\perp = (W_{I^\perp})$ . By construction, the set of  $m$  vectors  $Z^i$  in (4.6) and the set of  $\mathcal{N} - m$  vectors  $W^\perp$  must form a basis for  $\mathbb{C}^{\mathcal{N}}$ . The former is defined modulo the equivalence relation (4.8). Similarly, the latter is defined modulo the following equivalence relation

$$W_{I^\perp} \sim Z_I^{\underline{k}} E_{\underline{k}}^\perp + W_{I^\perp}^{\underline{K}} F_{\underline{K}}^\perp, \quad E_{\underline{k}}^\perp \in \mathbb{C}, \quad (F_{\underline{K}}^\perp) \in \text{GL}(\mathcal{N} - m, \mathbb{C}). \quad (4.13)$$

On an open subset of the space (4.7), the equivalence relation (4.8) can be used to choose

$$(Z_{I^\perp}) = \begin{pmatrix} \mathbb{1}_m \\ \zeta \end{pmatrix}, \quad (4.14)$$

where  $\zeta$  is a complex  $(\mathcal{N} - m) \times m$  matrix obeying the equation

$$\zeta^T \zeta = -\mathbb{1}_{\mathcal{N}-m}. \quad (4.15)$$

After that, the  $E$ -gauge freedom (4.13) allows us to choose

$$(W_{I^\perp}) = \begin{pmatrix} 0 \\ \omega \end{pmatrix}, \quad \omega \in \text{GL}(\mathcal{N} - m, \mathbb{C}). \quad (4.16)$$

Finally, the  $F$ -gauge freedom (4.13) allows us to choose

$$(W_{I^\perp}) = \begin{pmatrix} 0 \\ \mathbb{1}_{\mathcal{N}-m} \end{pmatrix}. \quad (4.17)$$

We see that the  $\mathcal{N} - m$  odd supertwistors  $\Upsilon^\perp$  do not add any new degrees of freedom.

Up to an equivalence transformation, we can always choose  $W^\perp$  to consist of two subsets  $W^\perp = \{\bar{Z}^i, R^\alpha\}$ , where  $\bar{Z}^i$  is the complex conjugate of  $Z^i$ , while  $2m$  vectors  $R^\alpha$  are real and orthogonal to  $Z^i$  and  $\bar{Z}^i$ ,

$$Z^i \cdot R^\alpha = \bar{Z}^i \cdot R^\alpha = 0, \quad i = 1, \dots, m, \quad \alpha = 1, \dots, \mathcal{N} - 2m. \quad (4.18)$$

Using the freedom to perform equivalence transformations (4.8), we can bring the Hermitian nonsingular matrix  $\bar{Z}^{\underline{i}} \cdot Z^{\underline{j}}$  to the form

$$\bar{Z}^{\underline{i}} \cdot Z^{\underline{j}} = \delta^{\underline{i}\underline{j}} . \quad (4.19)$$

If the null  $\mathcal{N}$ -vectors  $Z^{\underline{i}}$  are represented as a sum of their real and imaginary parts,  $Z^{\underline{i}} = \frac{1}{\sqrt{2}}(X^{\underline{i}} + iY^{\underline{i}})$ , then eq. (4.19) is equivalent to

$$X^{\underline{i}} \cdot X^{\underline{j}} = Y^{\underline{i}} \cdot Y^{\underline{j}} = \delta^{\underline{i}\underline{j}} , \quad X^{\underline{i}} \cdot Y^{\underline{j}} = 0 \quad \forall \underline{i}, \underline{j} = 1, \dots, m . \quad (4.20)$$

Using the freedom to perform equivalence transformations (4.8), we can also choose the real vector  $R^{\alpha}$  to form an orthonormal set,

$$R^{\alpha} \cdot R^{\beta} = \delta^{\alpha\beta} . \quad (4.21)$$

Now, the  $\mathcal{N}$  vectors  $\{X^{\underline{i}}, Y^{\underline{j}}, R^{\alpha}\}$  generate a group element  $g := (X_I^{\underline{i}}, Y_I^{\underline{j}}, R_I^{\alpha}) \in \text{SO}(\mathcal{N})$ .

### 4.3 Some special cases

For  $\mathcal{N} > 2$  and  $m = 1$  the internal manifold  $\mathbb{X}_1^{\mathcal{N}}$  proves to be a symmetric space,

$$\mathbb{X}_1^{\mathcal{N}} = \text{SO}(\mathcal{N}) / \text{SO}(\mathcal{N} - 2) \times \text{SO}(2) , \quad \mathcal{N} > 2 . \quad (4.22)$$

In the  $\mathcal{N} = 2$  case, the space  $\mathbb{X}_1^2$  reduces to just two points,  $Z_I^{(+)} = (1/\sqrt{2})(1, +i)$  and  $Z_I^{(-)} = (1/\sqrt{2})(1, -i)$ , which can be seen to correspond to the chiral and antichiral subspaces of  $\mathbb{M}^{3|4}$ .

In the case  $\mathcal{N} = 3$  and  $m = 1$ , to solve the null condition  $Z_I Z_I = 0$ , it is useful to replace the  $\text{SO}(3)$  index of  $Z_I$  by a pair of isospinor ones,

$$Z_I \rightarrow Z_i^j := \frac{i}{\sqrt{2}}(\vec{Z} \cdot \vec{\sigma})_i^j \equiv Z_I(\tau_I)_i^j , \quad Z_i^i = 0 , \quad (4.23)$$

with  $\vec{\sigma}$  the Pauli matrices. Then, the null condition  $Z_I Z_I = 0$  is solved by

$$Z^{ij} = v^i v^j , \quad v^i \in \mathbb{C}^2 \setminus \{0\} . \quad (4.24)$$

The equivalence relation (4.8) turns into

$$v^i \rightarrow \lambda v^i , \quad \lambda \in \mathbb{C} - \{0\} \quad (4.25)$$

and thus the internal space  $\mathbb{X}_1^3$  is  $\mathbb{C}P^1$ .

In the case  $\mathcal{N} = 4$  and  $m = 1$ , to solve the null condition  $Z_I Z_I = 0$ , it is useful to replace the  $\text{SO}(4) \cong (\text{SU}_L(2) \times \text{SU}_R(2))/\mathbb{Z}_2$  index of  $Z_I$  by a pair of different isospinor ones corresponding to the groups  $\text{SU}_L(2)$  and  $\text{SU}_R(2)$ ,

$$Z_I \rightarrow Z_i^{\bar{k}} := \frac{i}{\sqrt{2}}(\vec{Z} \cdot \vec{\sigma})_i^{\bar{k}} + \frac{1}{\sqrt{2}}Z_4\delta_i^{\bar{k}} \equiv Z_I(\tau_I)_i^{\bar{k}}. \quad (4.26)$$

Then, the null condition  $Z_I Z_I = 0$  is solved by

$$Z^{i\bar{k}} = v_L^i v_R^{\bar{k}} \quad (4.27)$$

for two non-zero isospinors  $v_L^i$  and  $v_R^{\bar{k}}$  transforming under the groups  $\text{SU}_L(2)$  and  $\text{SU}_R(2)$  respectively. The equivalence relation (4.8) turns into

$$v_L^i \rightarrow \lambda \mu v_L^i, \quad v_R^{\bar{k}} \rightarrow \frac{\lambda}{\mu} v_R^{\bar{k}}, \quad \lambda, \mu \in \mathbb{C} \setminus \{0\}. \quad (4.28)$$

We conclude that  $\mathbb{X}_1^4 = \mathbb{C}P^1 \times \mathbb{C}P^1$ .

In conclusion, consider the case  $\mathcal{N} = 4$  and  $m = 2$ . We have to deal with two linearly independent four-vector  $Z_{\pm} \in \mathbb{C}^4$  obeying the null conditions

$$Z_+ \cdot Z_+ = Z_- \cdot Z_- = Z_+ \cdot Z_- = 0. \quad (4.29)$$

A general solution to these conditions proves to be a sum of two partial solutions:

$$Z_{\pm}^{i\bar{k}} = w_{\pm}^i v_R^{\bar{k}}, \quad (w_+^i \ w_-^i) \in \text{GL}(2, \mathbb{C}), \quad v_R^{\bar{k}} \in \mathbb{C}^2 \setminus \{0\}; \quad (4.30a)$$

$$Z_{\pm}^{i\bar{k}} = v_L^i w_{\pm}^{\bar{k}}, \quad (w_+^{\bar{k}} \ w_-^{\bar{k}}) \in \text{GL}(2, \mathbb{C}), \quad v_L^i \in \mathbb{C}^2 \setminus \{0\}. \quad (4.30b)$$

For the family of first solutions, eq. (4.30a), the gauge freedom (4.8) can be used to choose

$$(w_+^i \ w_-^i) = \mathbb{1}_2. \quad (4.31)$$

After that we are still left with a residual gauge freedom of the form:

$$v_R^{\bar{k}} \sim \lambda v_R^{\bar{k}}, \quad \lambda \in \mathbb{C} \setminus \{0\}. \quad (4.32)$$

Therefore, the internal space in this case is  $\mathbb{C}P^1$ . The same consideration applies to the family of solutions (4.30b). We conclude that  $\mathbb{X}_2^4 = \mathbb{C}P^1 \cup \mathbb{C}P^1$ .

## 5 Superconformal Killing vector fields

For various studies of four-dimensional  $\mathcal{N} = 1$  superconformal field theories in superspace, the concept of superconformal Killing vectors [52, 53, 54, 55] has been shown to be indispensable, see e.g. [56, 57]. This concept has been generalised to  $\mathcal{N}$ -extended superconformal symmetry in three [58], four [51, 59] and six [60] dimensions. The five-dimensional case has been worked out in [11]. Here we elaborate on the salient properties of the 3D  $\mathcal{N}$ -extended superconformal Killing vectors in a form that is more close in spirit to the presentation given in [11].

### 5.1 $\mathcal{N}$ -extended superconformal Killing vector fields

Consider the 3D  $\mathcal{N}$ -extended Minkowski superspace  $\mathbb{M}^{3|2\mathcal{N}}$  parametrised by real bosonic and fermionic coordinates  $z^A = (x^a, \theta_I^\alpha)$ , where  $I = 1, \dots, \mathcal{N}$ . We recall that the spinor covariant derivatives  $D_\alpha^I$  obey the anticommutation relations

$$\{D_\alpha^I, D_\beta^J\} = 2i \delta^{IJ} (\gamma^m)_{\alpha\beta} \partial_m . \quad (5.1)$$

Superconformal transformations,  $z^A \rightarrow z^A + \delta z^A = z^A + \xi^A(z)$ , are generated by superconformal Killing vectors. By definition, a superconformal Killing vector

$$\xi = \xi^a(z) \partial_a + \xi_I^\alpha(z) D_\alpha^I \quad (5.2)$$

is a real vector field obeying the condition

$$[\xi, D_\alpha^I] \propto D_\beta^J \quad (5.3)$$

which is equivalent to

$$D_\alpha^I \xi^b = 2i \delta^{IJ} (\gamma^b)_{\alpha\beta} \xi_J^\beta . \quad (5.4)$$

Using this condition, it is a short calculation to show that  $\xi^a(x, \theta)$  is an ordinary conformal Killing vector (parametrically depending on  $\theta$ )

$$\partial_a \xi_b + \partial_b \xi_a = \frac{2}{3} \eta_{ab} \partial_c \xi^c . \quad (5.5)$$

An important implication of (5.4) is

$$\partial_{(\alpha\beta} \xi_{\gamma)}^I = 0 . \quad (5.6)$$

Direct calculations give

$$[\xi, D_\alpha^I] = -(D_\alpha^I \xi_\beta^J) D_\beta^J = \omega_\alpha{}^\beta(z) D_\beta^I + \Lambda^{IJ}(z) D_\alpha^J - \frac{1}{2} \sigma(z) D_\alpha^I, \quad (5.7)$$

where we have defined

$$\omega_{\alpha\beta} := -\frac{1}{\mathcal{N}} D_{(\alpha}^J \xi_{\beta)}^J = -\frac{1}{4} \partial^\gamma_{(\alpha} \xi_{\beta)\gamma}, \quad (5.8a)$$

$$\Lambda^{IJ} := -2 D_\alpha^{[I} \xi^{J]\alpha}, \quad (5.8b)$$

$$\sigma := \frac{1}{\mathcal{N}} D_\alpha^I \xi_I^\alpha = \frac{1}{3} \partial_a \xi^a. \quad (5.8c)$$

Here the parameters  $\omega_{\alpha\beta} = \omega_{\beta\alpha}$ ,  $\Lambda^{IJ} = -\Lambda^{JI}$  and  $\sigma$  correspond to  $z$ -dependent Lorentz,  $\text{SO}(\mathcal{N})$  and scale transformations. These transformation parameters are related to each other as follows:

$$D_\alpha^I \omega_{\beta\gamma} = \varepsilon_{\alpha(\beta} D_\gamma^I \sigma, \quad (5.9a)$$

$$D_\alpha^I \Lambda^{JK} = -2 \delta^{I[J} D_\alpha^{K]} \sigma. \quad (5.9b)$$

Making use of (5.8a) gives

$$\partial^{\alpha\beta} \omega_{\alpha\beta} = 0. \quad (5.10)$$

In conjunction with (5.9a), this identity implies that

$$D^{\alpha I} D_\alpha^I \sigma = 0. \quad (\text{no sum in } I) \quad (5.11)$$

## 5.2 $\mathcal{N} = 1$ superconformal Killing vector fields

In this subsection, we specialise the above results to the case  $\mathcal{N} = 1$ .

The  $\mathcal{N} = 1$  superconformal Killing vectors

$$\xi = \xi^a \partial_a + \xi^\alpha D_\alpha \quad (5.12)$$

are characterised by the property

$$[\xi, D_\alpha] = \omega_\alpha{}^\beta D_\beta - \frac{1}{2} \sigma D_\alpha, \quad (5.13)$$

where the  $z$ -dependent parameters of Lorentz ( $\omega_{\alpha\beta}$ ) and scale ( $\sigma$ ) transformations are

$$\omega_{\alpha\beta} := -D_{(\alpha} \xi_{\beta)} = -\frac{1}{4} \partial^\gamma_{(\alpha} \xi_{\beta)\gamma}, \quad (5.14a)$$

$$\sigma := D_\alpha \xi^\alpha = \frac{1}{3} \partial_a \xi^a. \quad (5.14b)$$

These parameters are related to each other by the relation

$$D_\alpha \omega_{\beta\gamma} = \varepsilon_{\alpha(\beta} D_\gamma) \sigma , \quad (5.15)$$

which implies

$$D^2 \sigma = 0 . \quad (5.16)$$

### 5.3 $\mathcal{N} = 2$ superconformal Killing vector fields

In the  $\mathcal{N} = 2$  case, it is useful to introduce a new basis for the spinor covariant derivatives. Instead of the covariant derivatives  $D_\alpha^I$ , with  $I = \mathbf{1}, \mathbf{2}$ , we introduce new operators  $\mathbb{D}_\alpha$  and  $\bar{\mathbb{D}}_\alpha$  defined as

$$\mathbb{D}_\alpha = \frac{1}{\sqrt{2}}(D_\alpha^{\mathbf{1}} - i D_\alpha^{\mathbf{2}}) , \quad \bar{\mathbb{D}}_\alpha = -\frac{1}{\sqrt{2}}(D_\alpha^{\mathbf{1}} + i D_\alpha^{\mathbf{2}}) . \quad (5.17)$$

They have definite  $\mathbf{U}(1)$  charges with respect to the group  $\mathbf{SO}(2) \in \mathbf{OSp}(2|2, \mathbb{R})$  and satisfy the anticommutation relations

$$\{\mathbb{D}_\alpha, \bar{\mathbb{D}}_\beta\} = -2i \partial_{\alpha\beta} , \quad \{\mathbb{D}_\alpha, \mathbb{D}_\beta\} = \{\bar{\mathbb{D}}_\alpha, \bar{\mathbb{D}}_\beta\} = 0 . \quad (5.18)$$

The superconformal Killing vector,  $\xi = \xi^a \partial_a + \xi_\alpha^I D_\alpha^I$ , takes the form

$$\xi = \xi^a \partial_a + \xi^\alpha \mathbb{D}_\alpha + \bar{\xi}_\alpha \bar{\mathbb{D}}^\alpha , \quad (5.19)$$

where the spinor  $\xi^\alpha$  and its complex conjugate  $\bar{\xi}^\alpha$  are defined by

$$\xi^\alpha = \frac{1}{\sqrt{2}}(\xi_1^\alpha + i \xi_2^\alpha) , \quad \bar{\xi}^\alpha = \frac{1}{\sqrt{2}}(\xi_1^\alpha - i \xi_2^\alpha) . \quad (5.20)$$

It is easy to see that  $\xi^\alpha$  is chiral,

$$\bar{\mathbb{D}}_\beta \xi^\alpha = 0 . \quad (5.21)$$

Our next task is to express the parameters (5.8a), (5.8b) and (5.8c) in terms of  $\xi^\alpha$  and its conjugate. Direct calculations give<sup>7</sup>

$$\omega_{\alpha\beta} = -\mathbb{D}_{(\alpha} \xi_{\beta)} = \bar{\mathbb{D}}_{(\alpha} \bar{\xi}_{\beta)} = -\frac{1}{4} \partial^\gamma_{(\alpha} \xi_{\beta)\gamma} , \quad (5.22a)$$

$$\Lambda^{IJ} = \varepsilon^{IJ} \Lambda , \quad \Lambda = \frac{i}{4} \left( \mathbb{D}_\gamma \xi^\gamma + \bar{\mathbb{D}}_\gamma \bar{\xi}^\gamma \right) , \quad (5.22b)$$

$$\sigma = \frac{1}{2} \left( \mathbb{D}_\gamma \xi^\gamma - \bar{\mathbb{D}}_\gamma \bar{\xi}^\gamma \right) = \frac{1}{3} \partial_a \xi^a . \quad (5.22c)$$

---

<sup>7</sup>Here the antisymmetric tensor  $\varepsilon^{IJ}$  is defined by  $\varepsilon^{\mathbf{12}} = 1$ .

If we define

$$\boldsymbol{\sigma} := \sigma + \mathrm{i} \Lambda \ , \quad \sigma = \frac{1}{2}(\boldsymbol{\sigma} + \bar{\boldsymbol{\sigma}}) \ , \quad (5.23)$$

then eq. (5.9b) is equivalent to the fact that  $\boldsymbol{\sigma}$  is chiral,

$$\bar{\mathbb{D}}_\alpha \boldsymbol{\sigma} = 0 \ . \quad (5.24)$$

It follows from eq. (5.11) that

$$\mathbb{D}^2 \boldsymbol{\sigma} = 0 \ . \quad (5.25)$$

Finally, the master relation (5.7) turns into

$$[\xi, \mathbb{D}_\alpha] = \omega_\alpha{}^\beta \mathbb{D}_\beta + \frac{1}{4}(\boldsymbol{\sigma} - 3\bar{\boldsymbol{\sigma}}) \mathbb{D}_\alpha \ . \quad (5.26)$$

## 5.4 $\mathcal{N} = 3$ superconformal Killing vector fields

As follows from eq. (5.8c), the case  $\mathcal{N} = 3$  is special in the sense that the superconformal transformations preserve the volume,  $\mathrm{d}^3x \mathrm{d}^6\theta$ , of Minkowski superspace  $\mathbb{M}^{3|6}$

$$(-1)^A D_A \xi^A = 0 \ , \quad D_A := (\partial_a, D_\alpha^I) \ . \quad (5.27)$$

This is similar to the four-dimensional  $\mathcal{N} = 2$  case, see e.g. [12].

It is useful to convert each  $\mathrm{SO}(3)$  vector index into a pair of two isospinor ones by the rule (4.23). The isospinor indices will be raised and lowered using the antisymmetric  $\mathrm{SU}(2)$  invariant tensor  $\varepsilon_{ij}$  and  $\varepsilon^{ij}$  (normalised as  $\varepsilon^{12} = \varepsilon_{21} = 1$ ). The rules for raising and lowering the isospinor indices are

$$\psi^i = \varepsilon^{ij} \psi_j \ , \quad \psi_i = \varepsilon_{ij} \psi^j \ . \quad (5.28)$$

In particular, associated with the matrices  $(\tau)_i{}^j$ , eq. (4.23), are the symmetric matrices  $(\tau_I)_{ij} = (\tau_I)_{ji}$  and  $(\tau_I)^{ij} = (\tau_I)^{ji}$  which are related to each other by complex conjugation:

$$\overline{(\tau_I)_{ij}} = (\tau_I)^{ij} \ . \quad (5.29)$$

If  $A_I$  and  $B_I$  are  $\mathrm{SO}(3)$  vectors and  $A_{ij}$  and  $B_{ij}$  the associated symmetric isotensors, then the dot-product reads as

$$A \cdot B := A_I B_I = A_{ij} B^{ij} \ . \quad (5.30)$$



Given an antisymmetric second-rank  $\text{SO}(3)$  tensor,  $\Lambda^{IJ} = -\Lambda^{JI}$ , its counterpart with isospinor indices  $\Lambda^{ijkl} = -\Lambda^{klij} = \Lambda^{IJ}(\tau_I)^{ij}(\tau_J)^{kl}$  can be decomposed as

$$\Lambda^{ijkl} = \varepsilon^{jl}\Lambda^{ik} + \varepsilon^{ik}\Lambda^{jl} , \quad \Lambda^{ij} = \Lambda^{ji} . \quad (5.31)$$

Note that the spinor covariant derivatives  $D_\alpha^{ij} = (\tau_I)^{ij}D_\alpha^I$  satisfy the algebra

$$\{D_\alpha^{ij}, D_\beta^{kl}\} = -2i\varepsilon^{i(k}\varepsilon^{l)j}\partial_{\alpha\beta} . \quad (5.32)$$

In terms of the superspace coordinates  $z^A = (x^a, \theta_{ij}^\alpha)$ , with  $\theta_{ij}^\alpha := (\tau_I)_{ij}\theta_I^\alpha$ , the explicit realisation of the covariant derivatives is

$$D_\alpha^{ij} = \frac{\partial}{\partial \theta_{ij}^\alpha} + i\theta_{ij}^\beta \partial_{\alpha\beta} . \quad (5.33)$$

The master relation (5.7) in the  $\text{SU}(2)$  notation takes the form

$$[\xi, D_\alpha^{ij}] = \omega_\alpha{}^\beta D_\beta^{ij} - \Lambda^i{}_k D_\alpha^{kj} - \Lambda^j{}_k D_\alpha^{ik} - \frac{1}{2}\sigma D_\alpha^{ij} , \quad (5.34)$$

where we now have  $(\xi_{ij}^\alpha = (\tau^I)_{ij}\xi_I^\alpha)$

$$\omega_{\alpha\beta} = -\frac{1}{3}D_{(\alpha}^{kl}\xi_{\beta)kl} = -\frac{1}{4}\partial_{(\alpha}{}^\gamma \xi_{\beta)\gamma} , \quad (5.35a)$$

$$\Lambda_{ij} = \frac{1}{4}D_{\alpha(i}^k \xi_{j)k}^\alpha , \quad (5.35b)$$

$$\sigma = \frac{1}{3}D_\alpha^{ij}\xi_{ij}^\alpha = \frac{1}{3}\partial^a \xi_a . \quad (5.35c)$$

The above parameters are related to each other by the following relations:

$$D_\alpha^{ij}\omega_{\beta\gamma} = \varepsilon_{\alpha(\beta}D_{\gamma)}^{ij}\sigma , \quad (5.36a)$$

$$D_\alpha^{ij}\Lambda^{kl} = \frac{1}{2}\varepsilon^{i(k}D_{\alpha}^{l)j}\sigma + \frac{1}{2}\varepsilon^{j(k}D_{\alpha}^{l)i}\sigma . \quad (5.36b)$$

According to the analysis of section 4, in the  $\mathcal{N} = 3$  case it natural to introduce a  $\text{SU}(2)$  complex isotwistor  $v^i$  which defines a null  $\text{SO}(3)$  vector  $Z_I = v^i v^j (\tau_I)_{ij}$ . It is useful to introduce two complex isospinor  $v^i$  and  $u^i$  which can be used to change basis for the isospinor indices e. g. by using the completeness relation

$$\delta_j^i = \frac{1}{(v, u)}(v^i u_j - v_j u^i) , \quad (v, u) := v^i u_i . \quad (5.37)$$

The choice of  $u_i$  is restricted only by the condition  $(v, u) \neq 0$ .

In the  $\{v, u\}$ -basis, the spinor covariant derivatives  $D_\alpha^{ij}$  turn into  $D_\alpha^{(2)}$ ,  $D_\alpha^{(0)}$  and  $D_\alpha^{(-2)}$  defined as follows

$$D_\alpha^{(2)} := v_i v_j D_\alpha^{ij}, \quad D_\alpha^{(0)} := \frac{1}{(v, u)} v_i u_j D_\alpha^{ij}, \quad D_\alpha^{(-2)} := \frac{1}{(v, u)^2} u_i u_j D_\alpha^{ij}, \quad (5.38)$$

where the superscript on  $D_\alpha^{(2)}$ ,  $D_\alpha^{(0)}$  and  $D_\alpha^{(-2)}$  indicates the degree of homogeneity in  $vs$ . The spinor covariant derivatives satisfy the algebra

$$\{D_\alpha^{(2)}, D_\beta^{(2)}\} = \{D_\alpha^{(-2)}, D_\beta^{(-2)}\} = 0, \quad (5.39a)$$

$$\{D_\alpha^{(2)}, D_\beta^{(-2)}\} = 2i\partial_{\alpha\beta}, \quad \{D_\alpha^{(0)}, D_\beta^{(0)}\} = -i\partial_{\alpha\beta}, \quad (5.39b)$$

$$\{D_\alpha^{(2)}, D_\beta^{(0)}\} = \{D_\alpha^{(-2)}, D_\beta^{(0)}\} = 0. \quad (5.39c)$$

The derivatives  $D_\alpha^{(2)}$  can be used to define analyticity constraints on superfields.

In the basis for the covariant derivatives introduced, the superconformal Killing vector takes the form

$$\xi = \xi^a \partial_a + \xi^{(2)\alpha} D_\alpha^{(-2)} - 2\xi^{(0)\alpha} D_\alpha^{(0)} + \xi^{(-2)\alpha} D_\alpha^{(2)}, \quad (5.40)$$

where we have introduced the spinor components

$$\xi^{(2)\alpha} := v^i v^j \xi_{ij}^\alpha, \quad \xi^{(0)\alpha} := \frac{1}{(v, u)} v^i u^j \xi_{ij}^\alpha, \quad \xi^{(-2)\alpha} := \frac{1}{(v, u)^2} u^i u^j \xi_{ij}^\alpha. \quad (5.41)$$

Note that the  $\xi^{(2)\alpha}$  satisfies the analyticity condition

$$D_\beta^{(2)} \xi^{(2)\alpha} = 0. \quad (5.42)$$

The covariant derivatives  $D_\alpha^{(2)}$  satisfy the master relation

$$\left[ \xi - \Lambda^{(2)} \boldsymbol{\partial}^{(-2)}, D_\alpha^{(2)} \right] = \omega_\alpha^\beta D_\beta^{(2)} - \left( 2\Sigma - \frac{1}{2}\sigma \right) D_\alpha^{(2)}. \quad (5.43)$$

Here we have introduced the scalar superfields

$$\Lambda^{(2)} := v_i v_j \Lambda^{ij}, \quad \Lambda^{(0)} := \frac{1}{(v, u)} v_i u_j \Lambda^{ij}, \quad \Sigma := \frac{1}{2}\sigma + \Lambda^{(0)}, \quad (5.44)$$

and used the isotwistor derivatives

$$\boldsymbol{\partial}^{(-2)} = \frac{1}{(v, u)} u^i \frac{\partial}{\partial v^i}, \quad \boldsymbol{\partial}^{(2)} = (v, u) v^i \frac{\partial}{\partial u^i}. \quad (5.45)$$

One can check that

$$D_\gamma^{(0)} \xi^{(2)\gamma} = -D_\gamma^{(2)} \xi^{(0)\gamma} = -2\Lambda^{(2)}. \quad (5.46)$$

The following important relations can be easily derived

$$D_\alpha^{(2)}\Lambda^{(2)} = D_\alpha^{(2)}\Sigma = 0 \ , \quad \boldsymbol{\partial}^{(2)}\Sigma = \Lambda^{(2)} \ . \quad (5.47)$$

For later use, it is important to note that the volume-preservation identity eq. (5.27) can be rewritten in the form

$$\partial_a \xi^a - D_\alpha^{(-2)} \xi^{\alpha(2)} + 2D_\alpha^{(0)} \xi^{\alpha(0)} - \boldsymbol{\partial}^{(-2)} \Lambda^{(2)} = 2\Sigma \ . \quad (5.48)$$

## 5.5 $\mathcal{N} = 4$ superconformal Killing vector fields

In complete analogy with  $\mathcal{N} = 3$  supersymmetry, in the  $\mathcal{N} = 4$  case it is also useful to convert the  $\text{SO}(4)$  vector indices into pairs of isospinor ones. This is achieved by making use of the matrices  $(\tau_I)_i^{\bar{k}}$  through the conversion rule (4.26). The isospinor indices  $i$  and  $\bar{k}$  respectively transform under  $\text{SU}_L(2)$  and  $\text{SU}_R(2)$  of  $\text{SO}(4) \cong (\text{SU}_L(2) \times \text{SU}_R(2))/\mathbb{Z}_2$ . By using the antisymmetric invariant tensors  $\varepsilon_{ij}, \varepsilon^{ij}$  and  $\varepsilon^{\bar{k}\bar{l}}, \varepsilon_{\bar{k}\bar{l}}$ , we will raise and lower the  $\text{SU}(2)$ s indices according to the rule

$$\psi^i = \varepsilon^{ij} \psi_j \ , \quad \psi_i = \varepsilon_{ij} \psi^j \ ; \quad (5.49a)$$

$$\chi_{\bar{k}} = \varepsilon_{\bar{k}\bar{l}} \chi^{\bar{l}} \ , \quad \chi^{\bar{k}} = \varepsilon^{\bar{k}\bar{l}} \chi_{\bar{l}} \ . \quad (5.49b)$$

Then, associated with the matrices  $(\tau_I)_i^{\bar{k}}$  there are the matrices  $(\tau_I)_{i\bar{k}} = \varepsilon_{\bar{k}\bar{l}} (\tau_I)_i^{\bar{l}}$  and  $(\tau_I)^{i\bar{k}} = \varepsilon^{ij} (\tau_I)_j^{\bar{k}}$  which are related to each other by complex conjugation:

$$\overline{(\tau_I)_{i\bar{k}}} = (\tau_I)^{i\bar{k}} \ . \quad (5.50)$$

Given two  $\text{SO}(4)$  vectors  $A_I$  and  $B_I$  and  $A_{i\bar{k}}$  and  $B_{i\bar{k}}$  their associated isotensors, the dot-product reads as

$$A \cdot B := A_I B_I = A_{i\bar{k}} B^{i\bar{k}} \ . \quad (5.51)$$

It follows that in the isospinor notation the covariant derivatives  $D_\alpha^{i\bar{k}} = (\tau_I)^{i\bar{k}} D_\alpha^I$  satisfies the algebra

$$\{D_\alpha^{i\bar{k}}, D_\beta^{j\bar{l}}\} = 2i\varepsilon^{ij}\varepsilon^{\bar{k}\bar{l}}\partial_{\alpha\beta} \ . \quad (5.52)$$

In terms of the superspace coordinates  $z^A = (x^a, \theta_{k\bar{l}}^\alpha)$ , with  $\theta_{k\bar{l}}^\alpha := (\tau_I)_{k\bar{l}} \theta_I^\alpha$ , the explicit realisation of the covariant derivatives is

$$D_\alpha^{k\bar{l}} = \frac{\partial}{\partial \theta_{k\bar{l}}^\alpha} + i\theta_{k\bar{l}}^\beta \partial_{\alpha\beta} \ . \quad (5.53)$$

An antisymmetric second-rank  $\text{SO}(4)$  tensor, like the superfield  $\Lambda^{IJ} = -\Lambda^{JI}$ , is transformed to the isotensor  $\Lambda^{i\bar{k}j\bar{l}} = -\Lambda^{j\bar{l}i\bar{k}} = \Lambda^{IJ}(\tau_I)^{i\bar{k}}(\tau_J)^{j\bar{l}}$ . This can be expressed in terms of its  $\text{SO}(4)$  self-dual and antiself-dual parts according to the following decomposition

$$\Lambda^{i\bar{k}j\bar{l}} = \varepsilon^{\bar{k}\bar{l}}\Lambda_L^{ij} + \varepsilon^{ij}\Lambda_R^{\bar{k}\bar{l}}, \quad \Lambda_L^{ij} = \Lambda_L^{ji}, \quad \Lambda_R^{\bar{k}\bar{l}} = \Lambda_R^{\bar{l}\bar{k}}. \quad (5.54)$$

The symmetric  $\Lambda_L^{ij}$  and  $\Lambda_R^{\bar{k}\bar{l}}$  parameters together with  $\omega_{\alpha\beta}$  and  $\sigma$  of (5.8a), (5.8c) are expressed in terms of the spinor  $\xi_{i\bar{k}}^\alpha = (\tau_I)_{i\bar{k}}\xi_I^\alpha$  as

$$\Lambda_L^{ij} = \frac{1}{4}D_{\bar{k}}^{\alpha(i}\xi_{\alpha}^{j)\bar{k}}, \quad \Lambda_R^{\bar{k}\bar{l}} = \frac{1}{4}D_{\alpha}^{i(\bar{k}}\xi_i^{\alpha\bar{l})}, \quad (5.55a)$$

$$\omega_{\alpha\beta} = -\frac{1}{4}D_{(\alpha}^{i\bar{k}}\xi_{\beta)i\bar{k}} = -\frac{1}{4}\partial_{(\alpha}{}^{\gamma}\xi_{\beta)\gamma}, \quad (5.55b)$$

$$\sigma = \frac{1}{4}D_{\alpha}^{i\bar{k}}\xi_{i\bar{k}}^{\alpha} = \frac{1}{3}\partial^a\xi_a. \quad (5.55c)$$

The above superfields turn out to be related to each other by the following equations

$$D_{\alpha}^{i\bar{k}}\omega_{\beta\gamma} = \varepsilon_{\alpha(\beta}D_{\gamma)}^{i\bar{k}}\sigma, \quad (5.56a)$$

$$D_{\alpha}^{i\bar{l}}\Lambda_L^{jk} = \varepsilon^{i(j}D_{\alpha}^{k)\bar{l}}\sigma, \quad D_{\alpha}^{i\bar{k}}\Lambda_R^{\bar{l}\bar{p}} = \varepsilon^{\bar{k}(\bar{l}}D_{\alpha}^{i\bar{p})}\sigma. \quad (5.56b)$$

Note also that the equation (5.7) in isospinor notation takes the form

$$[\xi, D_{\alpha}^{i\bar{k}}] = \omega_{\alpha}{}^{\beta}D_{\beta}^{i\bar{k}} + (\Lambda_L)^{ij}D_{\alpha j}^{\bar{k}} + (\Lambda_R)^{\bar{k}\bar{l}}D_{\alpha\bar{l}}^i - \frac{1}{2}\sigma D_{\alpha}^{i\bar{k}}. \quad (5.57)$$

By using two complex isotwistors, similarly to the analysis of subsection 5.4, we can change basis for the isospinor indices. For example, we introduce two left-isospinors  $v_L := (v^i)$  and  $u_L := (u^i)$ , which satisfy the very same relations as (5.37), and define new spinor covariant derivatives

$$D_{\alpha}^{(1)\bar{k}} := v_i D_{\alpha}^{i\bar{k}}, \quad D_{\alpha}^{(-1)\bar{k}} := \frac{1}{(v_L, u_L)} u_i D_{\alpha}^{i\bar{k}}. \quad (5.58)$$

Here the superscript on  $D_{\alpha}^{(1)\bar{k}}$  and  $D_{\alpha}^{(-1)\bar{k}}$  indicates the degree of homogeneity in  $vs$ . The spinor covariant derivatives satisfy the algebra

$$\{D_{\alpha}^{(1)\bar{k}}, D_{\beta}^{(1)\bar{l}}\} = \{D_{\alpha}^{(-1)\bar{k}}, D_{\beta}^{(-1)\bar{l}}\} = 0, \quad (5.59a)$$

$$\{D_{\alpha}^{(1)\bar{k}}, D_{\beta}^{(-1)\bar{l}}\} = -2i\varepsilon^{\bar{k}\bar{l}}\partial_{\alpha\beta}. \quad (5.59b)$$

In particular the  $D_{\alpha}^{(1)\bar{k}}$  derivatives represent a maximal anti-commuting subset of the  $D_{\alpha}^{i\bar{k}}$  derivatives and can be used to define consistent analyticity constraints.

In the covariant derivative basis just introduced, the superconformal Killing vector takes the form

$$\xi = \xi^a \partial_a - \xi^{(1)\alpha}_{\bar{k}} D_{\alpha}^{(-1)\bar{k}} + \xi^{(-1)\alpha}_{\bar{k}} D_{\alpha}^{(1)\bar{k}} , \quad (5.60)$$

where we have introduced the spinor components

$$\xi^{(1)\alpha}_{\bar{k}} := -v^i \xi_{i\bar{k}}^{\alpha} , \quad \xi^{(-1)\alpha}_{\bar{k}} := -\frac{1}{(v_L, u_L)} u^i \xi_{i\bar{k}}^{\alpha} . \quad (5.61)$$

Note that the  $\xi^{(1)\alpha}_{\bar{k}}$  superfield is constrained by the condition

$$D_{\beta(\bar{k}}^{(1)} \xi^{(1)\alpha}_{\bar{l})} = 0 . \quad (5.62)$$

The covariant derivatives  $D_{\alpha}^{(1)\bar{k}}$  satisfy the master relation

$$\left[ \xi - \Lambda_L^{(2)} \boldsymbol{\partial}_L^{(-2)} , D_{\alpha}^{(1)\bar{k}} \right] = \omega_{\alpha}^{\beta} D_{\beta}^{(1)\bar{k}} - (\varepsilon^{\bar{k}\bar{l}} \Sigma_L - \Lambda_R^{\bar{k}\bar{l}}) D_{\alpha\bar{l}}^{(1)} . \quad (5.63)$$

Here we have introduced the scalar superfields

$$\Lambda_L^{(2)} := v_i v_j \Lambda_L^{ij} , \quad \Lambda_L^{(0)} := \frac{1}{(v_L, u_L)} v_i u_j \Lambda_L^{ij} , \quad \Sigma_L := \frac{1}{2} \sigma + \Lambda_L^{(0)} . \quad (5.64)$$

One can derive the following relations between the previous parameters

$$D_{\alpha}^{(1)\bar{k}} \xi^{(1)\alpha}_{\bar{k}} = 4 \Lambda_L^{(2)} , \quad (5.65a)$$

$$D_{\alpha}^{(1)\bar{k}} \Lambda_L^{(2)} = D_{\alpha}^{(1)\bar{k}} \Sigma_L = 0 , \quad \boldsymbol{\partial}_L^{(2)} \Sigma_L = \Lambda_L^{(2)} . \quad (5.65b)$$

We conclude by giving the following equation

$$\partial_a \xi^a + D_{\alpha}^{(-1)\bar{k}} \xi^{(1)\alpha}_{\bar{k}} - \boldsymbol{\partial}_L^{(-2)} \Lambda_L^{(2)} = 2 \Sigma_L \quad (5.66)$$

which will be crucial in the formulation of a manifestly superconformal  $\mathcal{N} = 4$  action principle.

In complete analogy with the previous discussion, we also introduce right isotwistors  $v_R := (v^{\bar{k}})$  and  $u_R := (u_{\bar{k}})$  such that  $(v_R, u_R) := v^{\bar{k}} u_{\bar{k}} \neq 0$ , and use them to change basis for the right-isospinor indices. We then define new covariant derivatives

$$D_{\alpha}^{(1)i} := v_{\bar{k}} D_{\alpha}^{i\bar{k}} , \quad D_{\alpha}^{(-1)i} := \frac{1}{(v_R, u_R)} u_{\bar{k}} D_{\alpha}^{i\bar{k}} . \quad (5.67)$$

They satisfy anti-commutation relations analogous to eq. (5.59a) and (5.59b). Note that the spinor derivatives  $D_{\alpha}^{(1)i}$  represent a second maximal anti-commuting subset of the

operators  $D_\alpha^{i\bar{k}}$  and can be used to define a new type of constrained superfields. One can then rewrite the superconformal Killing vector in the  $D_\alpha^{(1)i}$  basis, and all the results in (5.60)–(5.66) carry over with the only modification that the left sector is changed everywhere with the right one.

The existence of the two independent sets of anti-commuting covariant derivatives  $D_\alpha^{(1)\bar{k}}$  and  $D_\alpha^{(1)i}$  is crucial to build the two type of hypermultiplets which describe matter in the  $\mathcal{N} = 4$  case. We will come back to this points in section 8.

## 6 $\mathcal{N} = 1$ and $\mathcal{N} = 2$ superconformal sigma-models

As a warm-up exercise in using the formalism of superconformal Killing vectors, in this section we construct the most general  $\mathcal{N} = 1, 2$  superconformal sigma-models.

### 6.1 $\mathcal{N} = 1$ superconformal sigma-models

To generate  $\mathcal{N} = 1$  superconformal actions, we need real scalar densities with the following transformation law

$$\delta\mathcal{L} = -\xi\mathcal{L} - 2\sigma\mathcal{L} , \quad (6.1)$$

with respect to the superconformal group. Given such a Lagrangian  $\mathcal{L}$ , the action

$$\int d^3x d^2\theta \mathcal{L} \quad (6.2)$$

is  $\mathcal{N} = 1$  superconformal.

Consider a general massless  $\mathcal{N} = 1$  sigma-model action

$$S = -\frac{1}{2} \int d^3x d^2\theta g_{\mu\nu}(\varphi) (D^\alpha \varphi^\mu) D_\alpha \varphi^\nu , \quad (6.3)$$

where  $g_{\mu\nu}(\varphi)$  is the metric on the target space. We wish to determine those restrictions on the target space geometry which guarantee the sigma-model to be superconformal. Without loss of generality, a superconformal transformation of  $\varphi^\mu$  can be chosen to be

$$\delta\varphi^\mu = -\xi\varphi^\mu - \frac{1}{2}\sigma\chi^\mu(\varphi) , \quad (6.4)$$

where  $\chi^m(\varphi)$  is a vector field on the target space. Using the properties of the  $\mathcal{N} = 1$  superconformal Killing vector fields, the variation of the action can be brought to the following form:

$$\begin{aligned} \delta S = & \frac{1}{2} \int d^3x d^2\theta g_{\mu\nu}(\varphi) (D^\alpha \varphi^\mu) (D_\alpha \varphi^\lambda) \left( \nabla_\lambda \chi^\nu(\varphi) - \delta_\lambda^\nu \right) \sigma \\ & + \frac{1}{2} \int d^3x d^2\theta g_{\mu\nu}(\varphi) (D^\alpha \varphi^\mu) \chi^\nu D_\alpha \sigma . \end{aligned} \quad (6.5)$$

In the case that  $D_\alpha \sigma = 0$  and  $\sigma$  is non-zero, the expression in the second line of (6.5) vanishes. Then, the remaining variation is equal to zero only if

$$\nabla_\mu \chi^\nu = \delta_\mu^\nu \quad \longleftrightarrow \quad \nabla_\mu \chi_\nu = g_{\mu\nu} . \quad (6.6)$$

We observe that  $\chi = \chi^\mu(\varphi) \partial_\mu$  should be a homothetic conformal Killing vector field such that  $\chi_\mu(\varphi)$  is the gradient of a function over the target space,

$$\chi_\mu(\varphi) = \partial_\mu f(\varphi) , \quad f(\varphi) = \frac{1}{2} g_{\mu\nu}(\varphi) \chi^\mu(\varphi) \chi^\nu(\varphi) . \quad (6.7)$$

The sigma-model target space is a Riemannian cone [8].

If the equations (6.6) and (6.7) hold, the variation (6.5) turns into

$$\delta S = \frac{1}{2} \int d^3x d^2\theta (D^\alpha f) D_\alpha \sigma = -\frac{1}{2} \int d^3x d^2\theta f D^2 \sigma = 0 , \quad (6.8)$$

since  $D^2 \sigma = 0$ .

The action (6.3) can be generalised by adding a potential term

$$S = -\frac{1}{2} \int d^3x d^2\theta g_{\mu\nu}(\varphi) (D^\alpha \varphi^\mu) D_\alpha \varphi^\nu + i \int d^3x d^2\theta V(\varphi^\mu) , \quad (6.9)$$

for some scalar field  $V(\varphi)$  on the target space. If the target space is a cone, and  $V(\varphi)$  obeys the homogeneity condition

$$\chi^\mu(\varphi) V_\mu(\varphi) = 4V(\varphi) , \quad (6.10)$$

then the action (6.9) is  $\mathcal{N} = 1$  superconformal.

## 6.2 $\mathcal{N} = 2$ superconformal sigma-models

There are two simple constructions to generate  $\mathcal{N} = 2$  superconformal actions. First, given a real superfield  $\mathcal{L}$  transforming by the rule

$$\delta \mathcal{L} = -\xi \mathcal{L} - \frac{1}{2} (\boldsymbol{\sigma} + \bar{\boldsymbol{\sigma}}) \mathcal{L} = -\partial_a (\xi^a \mathcal{L}) + \mathbb{D}_\alpha (\xi^\alpha \mathcal{L}) + \bar{\mathbb{D}}^\alpha (\bar{\xi}_\alpha \mathcal{L}) , \quad (6.11)$$

the functional

$$\int d^3x d^2\theta d^2\bar{\theta} \mathcal{L} \quad (6.12)$$

is  $\mathcal{N} = 2$  superconformal. Secondly, given a chiral superfield  $\mathcal{L}_c$ ,  $\bar{\mathbb{D}}_\alpha \mathcal{L}_c = 0$ , with the superconformal transformation

$$\delta \mathcal{L}_c = -\xi \mathcal{L}_c - 2\sigma \mathcal{L}_c = -\partial_a(\xi^a \mathcal{L}_c) + \mathbb{D}_\alpha(\xi^\alpha \mathcal{L}_c) , \quad (6.13)$$

the functional

$$\int d^3x d^2\theta \mathcal{L}_c \quad (6.14)$$

is  $\mathcal{N} = 2$  superconformal.

Consider the general  $\mathcal{N} = 2$  supersymmetric sigma-model [68]

$$S = \int d^3x d^2\theta d^2\bar{\theta} K(\Phi^I, \bar{\Phi}^{\bar{J}}) , \quad \bar{\mathbb{D}}_\alpha \Phi^I = 0 , \quad (6.15)$$

where  $K$  is the Kähler potential of a Kähler manifold  $\mathcal{M}$ . As usual, we denote by  $g_{I\bar{J}}(\Phi, \bar{\Phi})$  the Kähler metric on  $\mathcal{M}$ . Our goal is to determine those restrictions on the target space geometry which guarantee the sigma-model to be superconformal. Since (6.15) is a special  $\mathcal{N} = 1$  supersymmetric nonlinear sigma-model, it is  $\mathcal{N} = 1$  superconformal if the target space possesses a homothetic conformal Killing vector field  $\chi^\mu = (\chi^I, \bar{\chi}^{\bar{J}})$  such that  $\chi_\mu(\Phi)$  is the gradient of a function over the target space. The relations (6.6) and (6.7) turn into

$$\nabla_I \chi^J = \delta_I^J , \quad \bar{\nabla}_{\bar{I}} \chi^J = \bar{\partial}_{\bar{I}} \chi^J = 0 \quad (6.16a)$$

$$\chi_I := g_{I\bar{J}} \bar{\chi}^{\bar{J}} = \partial_I K , \quad g_{I\bar{J}} = \partial_I \bar{\partial}_{\bar{J}} K , \quad (6.16b)$$

where  $K$  can be chosen to be

$$K = g_{I\bar{J}} \chi^I \bar{\chi}^{\bar{J}} . \quad (6.17)$$

In accordance with (6.16a), the homothetic conformal Killing vector field is holomorphic,  $\chi^I = \chi^I(\Phi)$ . The target space  $\mathcal{M}$  is called a Kählerian cone [8].

The action (6.15) is invariant under the  $\mathcal{N} = 2$  superconformal transformations (compare with the 4D  $\mathcal{N} = 1$  case [14])

$$\delta \Phi^I = -\xi \Phi^I - \frac{1}{2} \sigma \chi^I(\Phi) , \quad (6.18)$$



with  $\xi$  an arbitrary  $\mathcal{N} = 2$  superconformal Killing vector. This follows from the identity

$$\chi^I(\Phi)K_I(\Phi, \bar{\Phi}) = K(\Phi, \bar{\Phi}) . \quad (6.19)$$

The sigma-model (6.15) can be generalised by including a superpotential.

$$S = \int d^3x d^2\theta d^2\bar{\theta} K(\Phi^I, \bar{\Phi}^{\bar{J}}) + \left\{ \int d^3x d^2\theta W(\Phi^I) + \text{c.c.} \right\} , \quad (6.20)$$

with  $W(\Phi)$  a holomorphic field on the target space. If  $\mathcal{M}$  is a Kählerian cone and  $W(\Phi)$  obeys the homogeneity condition

$$\chi^I(\Phi)W_I(\Phi) = 4W(\Phi) , \quad (6.21)$$

the sigma-model under consideration is  $\mathcal{N} = 2$  superconformal.

Local complex coordinates,  $\Phi^I$ , on  $\mathcal{M}$  can be chosen in such a way that  $\chi^I = \Phi^I$ . Then  $K(\Phi^I, \bar{\Phi}^{\bar{J}})$  obeys the following homogeneity condition:

$$\Phi^I \frac{\partial}{\partial \Phi^I} K(\Phi, \bar{\Phi}) = K(\Phi, \bar{\Phi}) . \quad (6.22)$$

Supersymmetric nonlinear sigma-models can also be generated from self-couplings of vector multiplets. Consider a dynamical system of several Abelian vector multiplets realised in terms of gauge-invariant real field strengths  $G^i$ , with  $i = 1, \dots, n$ , constrained as follows [33]:

$$\mathbb{D}^2 G^i = \bar{\mathbb{D}}^2 G^i = 0 , \quad i = 1, \dots, n . \quad (6.23)$$

Dynamics of the vector multiplets can be described by an action

$$\int d^3x d^2\theta d^2\bar{\theta} L(G^i) . \quad (6.24)$$

The constraints (6.23) uniquely fix the superconformal transformation of  $G^i$ :

$$\delta G^i = -\xi G^i - \frac{1}{2}(\sigma + \bar{\sigma})G^i . \quad (6.25)$$

Therefore, the action (6.24) is superconformal if the Lagrangian  $L(G^i)$  is a homogeneous function of the  $n$  variables  $G^i$  of degree one,

$$G^i L_i(G) = L(G) . \quad (6.26)$$

In the case of a single vector multiplet,  $n = 1$ , there is a unique superconformal model generated by a Lagrangian  $L(G) \propto -G \ln G$ . It describes an improved vector multiplet [33]. Such a model is generated as a low-energy effective action in quantum 3D  $\mathcal{N} = 2$  supersymmetric Yang-Mills theories [61].

## 7 Off-shell $\mathcal{N} = 3$ superconformal sigma-models

In this section we develop an efficient formalism to generate off-shell  $\mathcal{N} = 3$  superconformal sigma-models. It should be remarked that 3D  $\mathcal{N} = 3$  supersymmetry is quite interesting in its own right, since it can not be obtained by naive dimensional reduction from higher dimensions.

### 7.1 $\mathcal{N} = 3$ superconformal projective multiplets

We start with defining a family of off-shell  $\mathcal{N} = 3$  superconformal multiplets living in  $\mathcal{N} = 3$  projective superspace

$$\mathbb{M}^{3|6} \times \mathbb{C}P^1 . \quad (7.1)$$

The definition of such supermultiplets as well as the superconformal action principle (to be introduced in the next subsection) make use of the spinor covariant derivatives  $D_\alpha^{(2)}$ ,  $D_\alpha^{(0)}$  and  $D_\alpha^{(-2)}$  introduced earlier, eq. (5.38).

A superconformal projective multiplet of *integer* weight  $n$ ,  $Q^{(n)}(z, v)$ , is a superfield that lives on  $\mathbb{M}^{3|6}$ , is holomorphic with respect to the isotwistor variables  $v^i$  on an open domain of  $\mathbb{C}^2 \setminus \{0\}$  such that the following conditions hold:

(i) it obeys the analyticity constraints

$$D_\alpha^{(2)} Q^{(n)} = 0 ; \quad (7.2)$$

(ii) it is a homogeneous function of  $v^i$  of degree  $n$ , that is

$$Q^{(n)}(z, cv) = c^n Q^{(n)}(z, v) , \quad c \in \mathbb{C}^* := \mathbb{C} \setminus \{0\} ; \quad (7.3)$$

(iii) it possesses the superconformal transformation law:

$$\delta Q^{(n)} = - \left( \xi - \Lambda^{(2)} \boldsymbol{\partial}^{(-2)} \right) Q^{(n)} - n \Sigma Q^{(n)} . \quad (7.4)$$

As a consequence of eqs. (5.43) and (5.47), the variation  $\delta Q^{(n)}$  is analytic. By construction,  $Q^{(n)}$  is independent of the auxiliary isotwistor  $u_i$ ,

$$\frac{\partial}{\partial u^i} Q^{(n)} = 0 \quad \Longleftrightarrow \quad \boldsymbol{\partial}^{(2)} Q^{(n)} = 0 . \quad (7.5)$$

Eq. (7.3) implies that  $\delta Q^{(n)}$  is also independent of  $u_i$ ,

$$\boldsymbol{\partial}^{(2)} \delta Q^{(n)} = 0 , \quad (7.6)$$

although separate contributions to the right-hand side of (7.4) involve  $u_i$ .

As is clear from the above consideration, the isotwistor  $v^i \in \mathbb{C}^2 \setminus \{0\}$  is defined modulo the equivalence relation  $v^i \sim c v^i$ , with  $c \in \mathbb{C}^*$ , hence it parametrises  $\mathbb{CP}^1$ . Therefore, the projective multiplets live in  $\mathbb{M}^{3|6} \times \mathbb{CP}^1$ .

The definition given above is similar to that of 4D  $\mathcal{N} = 2$  superconformal projective multiplets [12]. The main difference is that the operators  $D_\alpha^{(2)}$  in the analyticity constraints (7.2) are quadratic in  $v^i$ , while the corresponding 4D operators are linear in  $v^i$ .

Given a superconformal weight- $n$  multiplet  $Q^{(n)}(v^i)$ , its *smile conjugate*,<sup>8</sup>  $\check{Q}^{(n)}(v^i)$ , is defined by

$$Q^{(n)}(v^i) \longrightarrow \bar{Q}^{(n)}(\bar{v}_i) \longrightarrow \bar{Q}^{(n)}(\bar{v}_i \rightarrow -v_i) =: \check{Q}^{(n)}(v^i) , \quad (7.7)$$

with  $\bar{Q}^{(n)}(\bar{v}_i) := \overline{Q^{(n)}(v^i)}$  the complex conjugate of  $Q^{(n)}(v^i)$ , and  $\bar{v}_i$  the complex conjugate of  $v^i$ . One can show that  $\check{Q}^{(n)}(v)$  is a superconformal weight- $n$  multiplet, unlike the complex conjugate of  $Q^{(n)}(v)$ . One can also check that

$$\check{\check{Q}}^{(n)}(v) = (-1)^n Q^{(n)}(v) . \quad (7.8)$$

Therefore, if  $n$  is even, one can define real isotwistor superfields,  $\check{Q}^{(2m)}(v) = Q^{(2m)}(v)$ .

Consider a superconformal Killing vector  $\xi_K$  that obeys the conditions

$$\Lambda_{ij}(z) = \sigma(z) = 0 , \quad (7.9)$$

with  $\Lambda_{ij}(z)$  and  $\sigma(z)$  defined in eqs. (5.35b) and (5.35b), respectively. It is called a  $\mathcal{N} = 3$  Killing vector, for the set of all such vectors can be seen to form a superalgebra isomorphic to the  $\mathcal{N} = 3$  super-Poincaré algebra. In the super-Poincaré case, the transformation law (7.4) reduces to the universal (weight-independent) form:

$$\delta Q^{(n)} = -\xi_K Q^{(n)} . \quad (7.10)$$

If we are interested in general  $\mathcal{N} = 2$  supersymmetric (i.e. super-Poincaré invariant) theories, not necessarily superconformal ones, projective multiplets should be defined by the relations (7.2), (7.3) and (7.10).

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<sup>8</sup>The smile conjugation is the real structure introduced in [20, 18, 15, 33].

## 7.2 $\mathcal{N} = 3$ superconformal action

Consider a real weight-2 projective multiplet  $\mathcal{L}^{(2)}(z, v)$ . We can associate with  $\mathcal{L}^{(2)}$  the following functional

$$S = \frac{1}{8\pi} \oint_{\gamma} v_i dv^i \int d^3x (D^{(-2)})^2 (D^{(0)})^2 \mathcal{L}^{(2)}(z, v) \Big|_{\theta=0} . \quad (7.11)$$

Here the line integral is carried out over a closed contour,  $\gamma = \{v_i(t)\}$ , in  $\mathbb{CP}^1$ . The covariant derivatives  $D_{\alpha}^{(-2)}$  and  $D_{\alpha}^{(0)}$  in (7.11) depend on a constant ( $t$ -independent) isotwistor  $u_i$ , in accordance with eq. (5.38), which is subject to the only condition that  $v(t)$  and  $u$  form a linearly independent basis at each point of the contour  $\gamma$ , that is  $(v(t), u) \neq 0$ . The functional is actually independent of  $u$ , since it is invariant under arbitrary projective transformations of the form

$$(u_i, v_i) \rightarrow (u_i, v_i) R, \quad R = \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \in \text{GL}(2, \mathbb{C}) . \quad (7.12)$$

This invariance follows from the following three observations. First of all, an infinitesimal transformation  $\delta u_i = b v_i$  acts on the covariant derivatives in (7.11) as

$$\delta D_{\alpha}^{(-2)} = \frac{2b}{(v, u)} D_{\alpha}^{(0)}, \quad \delta D_{\alpha}^{(0)} = \frac{b}{(v, u)} D_{\alpha}^{(2)} . \quad (7.13)$$

The second observation is that  $(D^{(0)})^3 \propto \partial D^{(0)}$ . The third observation is the analyticity of  $\mathcal{L}^{(2)}$ , that is  $D_{\alpha}^{(2)} \mathcal{L}^{(2)} = 0$ .

The action (7.11) proves to be  $\mathcal{N} = 3$  superconformal. Indeed, making use of eq. (5.48), the superconformal transformation of  $\mathcal{L}^{(2)}$  can be rewritten in the form

$$\delta \mathcal{L}^{(2)} = -\partial_a \left( \xi^a \mathcal{L}^{(2)} \right) + D_{\alpha}^{(-2)} \left( \xi^{(2)\alpha} \mathcal{L}^{(2)} \right) - 2D_{\alpha}^{(0)} \left( \xi^{(0)\alpha} \mathcal{L}^{(2)} \right) + \boldsymbol{\partial}^{(-2)} \left( \Lambda^{(2)} \mathcal{L}^{(2)} \right) . \quad (7.14)$$

Here the first three terms on the right do not contribute to the variation of the action (7.11). It remains to show that the last term also produces no contribution to the variation of the action. To achieve this, we first point out the identity

$$(D^{(-2)})^2 \left[ \boldsymbol{\partial}^{(-2)}, (D^{(0)})^2 \right] = 0 . \quad (7.15)$$

Our second observation is that (compare with eq. (2.50) in [12])

$$\begin{aligned} (\dot{v}, v) (D^{(-2)})^2 (D^{(0)})^2 \boldsymbol{\partial}^{(-2)} \left( \Lambda^{(2)} \mathcal{L}^{(2)} \right) &= (u_i u_j D^{ij})^2 \frac{(\dot{v}, v)}{(v, u)^4} \boldsymbol{\partial}^{(-2)} \left( (D^{(0)})^2 \Lambda^{(2)} \mathcal{L}^{(2)} \right) \\ &= -\frac{d}{dt} \left( (D^{(-2)})^2 (D^{(0)})^2 \Lambda^{(2)} \mathcal{L}^{(2)} \right) , \end{aligned} \quad (7.16)$$

where  $(\dot{v}, v) dt = v_i dv^i$  is part of the line integral measure in (7.11). The result obtained is a special case of the following general property: if  $f^{(n)}(v)$  is a homogeneous function of  $v^i$  of degree  $n$ ,  $f^{(n)}(cv) = c^n f^{(n)}(v)$ , then

$$\frac{(\dot{v}, v)}{(v, u)^n} \partial^{(-2)} f^{(n)}(v) = -\frac{d}{dt} \left( \frac{f^{(n)}(v)}{(v, u)^n} \right) . \quad (7.17)$$

Since the line integral in (7.11) is a closed contour, we conclude that the action is invariant under the  $\mathcal{N} = 3$  superconformal transformations.

### 7.3 Projective multiplets in the north chart of two-sphere

Without loss of generality, we can assume that the integration contour  $\gamma$  in (7.11) does not pass through the “north pole”  $v_{\text{north}}^i \sim (0, 1)$  of  $\mathbb{CP}^1$ . It is then useful to introduce a complex (*inhomogeneous*) coordinate  $\zeta$  in the north chart,  $\mathbb{C}$ , of  $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$ :

$$v^i = v^1 (1, \zeta) , \quad \zeta := \frac{v^2}{v^1} , \quad i = 1, 2 \quad (7.18)$$

and consider the projective multiplets in this chart. Given a weight- $n$  projective superfield  $Q^{(n)}(z, v)$ , we can associate with it a new object  $Q^{[n]}(z, \zeta)$  defined as

$$Q^{(n)}(z, v) \longrightarrow Q^{[n]}(z, \zeta) \propto Q^{(n)}(z, v) , \quad \frac{\partial}{\partial \bar{\zeta}} Q^{[n]} = 0 . \quad (7.19)$$

The explicit form of  $Q^{[n]}(z, \zeta)$  depends on the multiplet under consideration. The  $Q^{[n]}(z, \zeta)$  can be represented by a Laurent series

$$Q^{[n]}(z, \zeta) = \sum \zeta^k Q_k(z) , \quad (7.20)$$

with  $Q_k(z)$  some ordinary  $\mathcal{N} = 3$  superfields.

The covariant derivative  $D_\alpha^{(2)}$  appearing in (7.2) can now be represented as

$$\begin{aligned} D_\alpha^{(2)} &= (v^1)^2 D_\alpha^{[2]} , & D_\alpha^{[2]}(\zeta) &= D_\alpha^{22} - 2\zeta D_\alpha^{12} + \zeta^2 D_\alpha^{11} \\ & & &= -\bar{\mathbb{D}}_\alpha - 2\zeta D_\alpha^{12} + \zeta^2 \mathbb{D}_\alpha , \end{aligned} \quad (7.21)$$

where we have introduced the spinor covariant derivatives for  $\mathcal{N} = 2$  superspace,  $\mathbb{D}_\alpha := D_\alpha^{11}$  and  $\bar{\mathbb{D}}_\alpha := -D_\alpha^{22}$ , see also subsection C.1. The analyticity conditions (7.2) turn into

$$D_\alpha^{[2]}(\zeta) Q^{[n]}(\zeta) = 0 . \quad (7.22)$$

These constraints have a simple and, at the same time, very important interpretation: for all the component  $\mathcal{N} = 3$  superfields  $Q_k$  appearing in the series (7.20), their dependence on  $\theta_{12}^\alpha$  is uniquely determined, according to (7.22), in terms of their dependence on the Grassmann variables of  $\mathcal{N} = 2$  superspace. In other words, all information about the projective multiplet is encoded in its  $\mathcal{N} = 2$  projection

$$Q^{[n]}(\zeta)| := Q^{[n]}(\zeta)|_{\theta_{12}=0} . \quad (7.23)$$

We now give two examples of superconformal projective multiplet. An off-shell hypermultiplet can be described in term of the so-called *arctic* weight- $n$  multiplet  $\Upsilon^{(n)}(z, v)$  which is defined to be holomorphic in the north chart of  $\mathbb{CP}^1$ ,

$$\Upsilon^{(n)}(z, v) = (v^1)^n \Upsilon^{[n]}(z, \zeta) , \quad \Upsilon^{[n]}(z, \zeta) = \sum_{k=0}^{\infty} \Upsilon_k(z) \zeta^k , \quad (7.24)$$

and its smile-conjugate *antarctic* multiplet  $\check{\Upsilon}^{(n)}(z, v)$ ,

$$\check{\Upsilon}^{(n)}(z, v) = (v^1 \zeta)^n \check{\Upsilon}^{[n]}(z, \zeta) , \quad \check{\Upsilon}^{[n]}(z, \zeta) = \sum_{k=0}^{\infty} \bar{\Upsilon}_k(z) \frac{(-1)^k}{\zeta^k} . \quad (7.25)$$

The pair  $\Upsilon^{[n]}(\zeta)$  and  $\check{\Upsilon}^{[n]}(\zeta)$  constitute the so-called polar weight- $n$  multiplet. The analyticity constraints (7.22) imply that  $\Upsilon_0$  is  $\mathcal{N} = 2$  chiral and  $\Sigma_1$  is  $\mathcal{N} = 2$  complex linear,

$$\bar{\mathbb{D}}_\alpha \Upsilon_0 = 0 , \quad \bar{\mathbb{D}}^2 \Upsilon_1 = 0 , \quad (7.26)$$

while the other components  $\Upsilon_2, \Upsilon_3, \dots$ , are unconstrained complex  $\mathcal{N} = 2$  superfields.

Our second example is the so-called real *tropical* multiplet  $U^{(2n)}(z, v)$  of weight  $n$  defined by

$$U^{(2n)}(z, v) = (\mathbf{i} v^1 v^2)^n U^{[2n]}(z, \zeta) = (v^1)^{2n} (\mathbf{i} \zeta)^n U^{[2n]}(z, \zeta) , \\ U^{[2n]}(z, \zeta) = \sum_{k=-\infty}^{\infty} U_k(z) \zeta^k , \quad \bar{U}_k = (-1)^k U_{-k} . \quad (7.27)$$

An example of such a multiplet with  $n = 2$  is the Lagrangian  $\mathcal{L}^{(2)}$  in (7.11). The case  $n = 0$  is used to describe a vector multiplet [17].

In the  $\mathcal{N} = 3$  supersymmetric action (7.11), the Lagrangian  $\mathcal{L}^{(2)}$  is a projective multiplet, and therefore it is fully determined by its  $\mathcal{N} = 2$  projection  $\mathcal{L}^{(2)}|_{\theta_{12}=0}$ . The action (7.11) can be expressed in terms of this projection. We recall that the integration contour

$\gamma$  in (7.11) is chosen to lie outside the “north pole”  $v_{\text{north}}^i \sim (0, 1)$  of  $\mathbb{CP}^1$ , which allows us to use the inhomogeneous complex coordinate,  $\zeta$ , defined by  $v^i = v^1(1, \zeta)$ . Since the action is independent of  $u_i$ , the latter can be chosen to be  $u_i = (1, 0)$ , such that  $(v, u) = v^1 \neq 0$ . We represent the Lagrangian in the form:

$$\mathcal{L}^{(2)}(z, v) = i v^1 v^2 \mathcal{L}(z, \zeta) = i(v^1)^2 \zeta \mathcal{L}(z, \zeta) , \quad \check{\mathcal{L}} = \mathcal{L} . \quad (7.28)$$

Using the analyticity conditions  $D_\alpha^{[2]}(\zeta) \mathcal{L}(z, \zeta) = 0$  allows us to rewrite (7.11) in the form:

$$S = \frac{1}{2\pi i} \oint_\gamma \frac{d\zeta}{\zeta} \int d^3x d^2\theta d^2\bar{\theta} \mathcal{L}(z, \zeta) \Big|_{\theta_{12}=0} . \quad (7.29)$$

Here the integration is carried out over the  $\mathcal{N} = 2$  superspace. The action is now formulated entirely in terms of  $\mathcal{N} = 2$  superfields. At the same time, by construction, it is off-shell  $\mathcal{N} = 3$  supersymmetric.

## 7.4 $\mathcal{N} = 3$ superconformal sigma-models

We consider a system of interacting weight-one arctic multiplets,  $\Upsilon^{(1)I}(z, v)$ , and their smile-conjugates,  $\check{\Upsilon}^{(1)\bar{I}}(z, v)$ , described by a Lagrangian<sup>9</sup> of the form [12]:

$$\mathcal{L}^{(2)}(\Upsilon^{(1)}, \check{\Upsilon}^{(1)}) = i K(\Upsilon^{(1)I}, \check{\Upsilon}^{(1)\bar{J}}) . \quad (7.30)$$

Here  $K(\Phi^I, \bar{\Phi}^{\bar{J}})$  is a real function of  $n$  complex variables  $\Phi^I$ , with  $I = 1, \dots, n$ , under the homogeneity condition

$$\Phi^I \frac{\partial}{\partial \Phi^I} K(\Phi, \bar{\Phi}) = K(\Phi, \bar{\Phi}) . \quad (7.31)$$

The function  $K(\Phi^I, \bar{\Phi}^{\bar{J}})$  can be interpreted as the Kähler potential of a *Kählerian cone*  $\mathcal{M}$  written in special complex coordinates in which the homothetic conformal Killing vector field  $\chi^I(\Phi)$  has the form  $\chi^I(\Phi) = \Phi^I$ , see subsection 6.2.

By construction, the action generated by the Lagrangian (7.30) is invariant under the  $\mathcal{N} = 3$  superconformal transformation

$$\delta \Upsilon^{(1)I} = - \left( \xi - \Lambda^{(2)} \boldsymbol{\partial}^{(-2)} \right) \Upsilon^{(1)I} - \Sigma \Upsilon^{(1)I} . \quad (7.32)$$

Keeping in mind the  $\mathcal{N} = 2$  superconformal transformation law, eq. (6.18), one could be tempted to put forward a different transformation law of the form:

$$\hat{\delta} \Upsilon^{(1)I} = - \left( \xi - \Lambda^{(2)} \boldsymbol{\partial}^{(-2)} \right) \Upsilon^{(1)I} - \Sigma \chi^I(\Upsilon^{(1)}) .$$

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<sup>9</sup>The action generated by the Lagrangian (7.30) is real due to (7.8).

However, such a transformation is inconsistent with the arctic multiplet structure on two grounds: (i) the second term in  $\hat{\delta}\Upsilon^{(1)I}$  is not a homogeneous function of  $v^i$  of first degree unless  $\chi^I(\Phi)$  is also homogenous of first degree, which is true if  $\chi^I(\Phi) \propto \Phi^I$ ; (ii) the variation  $\hat{\delta}\Upsilon^{(1)I}$  depends explicitly on the isotwistor  $u_i$  unless  $\chi^I \propto \Upsilon^{(1)I}$ . Off-shell  $\mathcal{N} = 3$  supersymmetry requires special complex coordinates for Kählerian cones.

There exists a more geometric formulation of the theory (7.30) described in detail in [13]. It is realised in terms of a single weight-one arctic multiplet  $\Upsilon^{(1)}$  and  $n-1$  weight-zero arctic multiplets  $\Xi^i$ . The corresponding Lagrangian is

$$K(\Upsilon^{(1)I}, \check{\Upsilon}^{(1)\bar{J}}) = \Upsilon^{(1)} \check{\Upsilon}^{(1)} \exp \left\{ \mathcal{K}(\Xi^i, \check{\Xi}^{\bar{j}}) \right\} , \quad (7.33)$$

where the original variables  $\Upsilon^{(1)I}$  are related to the new ones by a holomorphic reparametrisation. The arctic variables  $\Upsilon^{(1)}$  and  $\Xi^i$  parametrise a holomorphic line bundle over a Kähler-Hodge manifold with Kähler potential  $\mathcal{K}(\varphi^i, \bar{\varphi}^{\bar{j}})$ , see [13] for more details. We will not use this formulation in the present paper.

Once reformulated in  $\mathcal{N} = 2$  superspace, the  $\mathcal{N} = 3$  superconformal sigma-model action takes the form:

$$S = \frac{1}{2\pi i} \oint_{\gamma} \frac{d\zeta}{\zeta} \int d^3x d^2\theta d^2\bar{\theta} K(\Upsilon^I, \check{\Upsilon}^{\bar{J}}) . \quad (7.34)$$

Here the weight-one arctic multiplets  $\Upsilon^I(\zeta) \equiv \Upsilon^{[1]I}(\zeta)$  and their smile-conjugates  $\check{\Upsilon}^{\bar{I}}(\zeta) \equiv \check{\Upsilon}^{[1]\bar{I}}(\zeta)$  embrace an infinite number of ordinary  $\mathcal{N} = 2$  superfields.

$$\Upsilon^I(\zeta) = \sum_{k=0}^{\infty} \zeta^k \Upsilon_k^I = \Phi^I + \zeta \Sigma^I + O(\zeta^2) , \quad \bar{\mathbb{D}}_{\alpha} \Phi^I = 0 , \quad \bar{\mathbb{D}}^2 \Sigma^I = 0 , \quad (7.35a)$$

$$\check{\Upsilon}^{\bar{J}}(\zeta) = \sum_{n=0}^{\infty} (-\zeta)^{-n} \bar{\Upsilon}_n^{\bar{J}} . \quad (7.35b)$$

We recall that the components  $\Upsilon_2, \Upsilon_3, \dots$ , are complex unconstrained  $\mathcal{N} = 2$  superfields.

Our off-shell  $\mathcal{N} = 3$  superconformal sigma-model (7.34) is determined by a real function of  $n$  complex variables  $K(\Phi^I, \bar{\Phi}^{\bar{J}})$  which is arbitrary modulo the homogeneity condition (7.31). We thus have a powerful scheme to generate  $\mathcal{N} = 3$  superconformal sigma-models, and therefore hyperkähler cones.

One can consider a more general sigma-model action, than the superconformal theory (7.34), of the form [16]:

$$S = \frac{1}{2\pi i} \oint_{\gamma} \frac{d\zeta}{\zeta} \int d^3x d^2\theta d^2\bar{\theta} L(\Upsilon^I, \check{\Upsilon}^{\bar{J}}, \zeta) . \quad (7.36)$$



Here the Lagrangian is no longer required to obey any homogeneity condition and, moreover, can explicitly depend on  $\zeta$ . The action is no longer superconformal, but it is off-shell  $\mathcal{N} = 3$  supersymmetric. It is believed that (7.36) describes the most general  $\mathcal{N} = 3$  supersymmetric sigma-model, see [62] for more details.

## 7.5 Reformulation in terms of $\mathbf{N} = 2$ chiral superfields

The  $\mathcal{N} = 3$  superconformal sigma-model (7.34) involves the infinite set of auxiliary  $\mathcal{N} = 2$  superfields  $\Upsilon_2^I, \Upsilon_3^I, \dots$ , which are necessary to realise off-shell supersymmetry. In order to describe the theory in terms of the physical superfields  $\Phi^I$  and  $\Sigma^I$  only, all the auxiliary superfields have to be eliminated with the aid of the corresponding algebraic equations of motion

$$\oint \frac{d\zeta}{\zeta} \zeta^n \frac{\partial K(\Upsilon, \check{\Upsilon})}{\partial \Upsilon^I} = \oint \frac{d\zeta}{\zeta} \zeta^{-n} \frac{\partial K(\Upsilon, \check{\Upsilon})}{\partial \check{\Upsilon}^{\bar{J}}} = 0, \quad n \geq 2. \quad (7.37)$$

Upon elimination of the auxiliaries, which is a difficult problem, the superconformal symmetry becomes model-dependent and on-shell. The determination of its explicit form is a nontrivial technical problem. Fortunately, similar problems have been analysed in the case of 4D  $\mathcal{N} = 2$  superconformal sigma-models in [14] (building on earlier works [63, 64]). Here we can simply recycle the results of [14].

Upon elimination of the auxiliary superfields, the action (7.34) turns into

$$S[\Phi, \Sigma] = \int d^3x d^2\theta d^2\bar{\theta} \left\{ K(\Phi, \bar{\Phi}) + \mathcal{L}(\Phi, \bar{\Phi}, \Sigma, \bar{\Sigma}) \right\},$$

$$\mathcal{L}(\Phi, \bar{\Phi}, \Sigma, \bar{\Sigma}) = \sum_{n=1}^{\infty} \mathcal{L}_{I_1 \dots I_n \bar{J}_1 \dots \bar{J}_n}(\Phi, \bar{\Phi}) \Sigma^{I_1} \dots \Sigma^{I_n} \bar{\Sigma}^{\bar{J}_1} \dots \bar{\Sigma}^{\bar{J}_n}, \quad (7.38)$$

where  $\mathcal{L}_{I\bar{J}} = -g_{I\bar{J}}(\Phi, \bar{\Phi})$  and the coefficients  $\mathcal{L}_{I_1 \dots I_n \bar{J}_1 \dots \bar{J}_n}$ , for  $n > 1$ , are tensor functions of the Kähler metric  $g_{I\bar{J}}(\Phi, \bar{\Phi}) = \partial_I \partial_{\bar{J}} K(\Phi, \bar{\Phi})$ , the Riemann curvature  $R_{I\bar{J}K\bar{L}}(\Phi, \bar{\Phi})$  and its covariant derivatives. The function  $\mathcal{L}$  is characterised by the property

$$\Sigma^I \frac{\partial \mathcal{L}}{\partial \Sigma^I} = \bar{\Sigma}^{\bar{I}} \frac{\partial \mathcal{L}}{\partial \bar{\Sigma}^{\bar{I}}} \quad (7.39)$$

since the original model (7.34) is invariant under rigid  $\mathbf{U}(1)$  transformations

$$\Upsilon(\zeta) \mapsto \Upsilon(e^{i\alpha}\zeta) \iff \Upsilon_n(z) \mapsto e^{in\alpha} \Upsilon_n(z). \quad (7.40)$$

The Lagrangian  $\mathcal{L}(\Phi, \bar{\Phi}, \Sigma, \bar{\Sigma})$  in (7.38) obeys the homogeneity condition

$$\left( \Phi^I \frac{\partial}{\partial \Phi^I} + \Sigma^I \frac{\partial}{\partial \Sigma^I} \right) \mathcal{L}(\Phi, \bar{\Phi}, \Sigma, \bar{\Sigma}) = \mathcal{L}(\Phi, \bar{\Phi}, \Sigma, \bar{\Sigma}). \quad (7.41)$$

Even though the action (7.38) is formulated in terms of the physical superfields only, its Lagrangian  $K(\Phi, \bar{\Phi}) + L(\Phi, \bar{\Phi}, \Sigma, \bar{\Sigma})$  is not a hyperkähler potential, since the dynamical variable  $\Sigma$  is complex linear. As discussed in section 9 the Lagrangian coincides with the hyperkähler potential of the target space provided the theory is formulated in terms of  $\mathcal{N} = 2$  chiral superfields and their conjugates only. The theory has to be re-formulated in terms of chiral superfields by performing a duality transformation known as the generalised Legendre transform construction [16]. To construct a dual formulation of the theory (7.38), we consider the first-order action

$$S_{\text{first-order}} = \int d^3x d^2\theta d^2\bar{\theta} \left\{ K(\Phi, \bar{\Phi}) + \mathcal{L}(\Phi, \bar{\Phi}, \Sigma, \bar{\Sigma}) + \Psi_I \Sigma^I + \bar{\Psi}_{\bar{I}} \bar{\Sigma}^{\bar{I}} \right\}. \quad (7.42)$$

Here the tangent vector  $\Sigma^I$  is complex unconstrained, while the one-form  $\Psi_I$  is chiral,  $\bar{D}_{\dot{\alpha}} \Psi_I = 0$ . Eliminating  $\Sigma$ 's and their conjugates, by using their equations of motion

$$\frac{\partial}{\partial \Sigma^I} \mathcal{L}(\Phi, \bar{\Phi}, \Sigma, \bar{\Sigma}) + \Psi_I = 0, \quad (7.43)$$

leads to the dual action

$$S[\Phi, \Psi] = \int d^3x d^2\theta d^2\bar{\theta} \mathbb{K}(\Phi, \bar{\Phi}, \Psi, \bar{\Psi}), \quad (7.44)$$

where the Lagrangian has the form

$$\begin{aligned} \mathbb{K}(\Phi, \bar{\Phi}, \Psi, \bar{\Psi}) &= K(\Phi, \bar{\Phi}) + \mathcal{H}(\Phi, \bar{\Phi}, \Psi, \bar{\Psi}), \\ \mathcal{H}(\Phi, \bar{\Phi}, \Psi, \bar{\Psi}) &= \sum_{n=1}^{\infty} \mathcal{H}^{I_1 \dots I_n \bar{J}_1 \dots \bar{J}_n}(\Phi, \bar{\Phi}) \Psi_{I_1} \dots \Psi_{I_n} \bar{\Psi}_{\bar{J}_1} \dots \bar{\Psi}_{\bar{J}_n} \end{aligned} \quad (7.45)$$

and  $\mathcal{H}^{I\bar{J}}(\Phi, \bar{\Phi}) = g^{I\bar{J}}(\Phi, \bar{\Phi})$ . The function  $\mathcal{H}$  is characterised by the following homogeneity properties:

$$\Psi_I \frac{\partial \mathcal{H}}{\partial \Psi_I} = \bar{\Psi}_{\bar{I}} \frac{\partial \mathcal{H}}{\partial \bar{\Psi}_{\bar{I}}}, \quad (7.46a)$$

$$\left( \Phi^I \frac{\partial}{\partial \Phi^I} + \Psi_I \frac{\partial}{\partial \Psi_I} \right) \mathcal{H} = \mathcal{H}. \quad (7.46b)$$

The derivation of the above results is similar to the 4D  $\mathcal{N} = 2$  case [14].

The Lagrangian  $\mathbb{K}(\Phi, \bar{\Phi}, \Psi, \bar{\Psi})$  in (7.44) is the Kähler potential of a hyperkähler cone. The action is still  $\mathcal{N} = 3$  superconformal, however the superconformal transformations form a closed algebra only on the mass shell. To describe the symmetries of the model, we introduce the condensed notation

$$\phi^a := (\Phi^I, \Psi_I), \quad \bar{\phi}^{\bar{a}} = (\bar{\Phi}^{\bar{I}}, \bar{\Psi}_{\bar{I}}), \quad (7.47)$$

as well as the standard symplectic matrix  $\mathbb{J} = (\mathbb{J}^{ab})$ , its inverse  $\mathbb{J}^{-1} = (-\mathbb{J}_{ab})$  and their complex conjugates,

$$\mathbb{J}^{ab} = \mathbb{J}^{\bar{a}\bar{b}} = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}, \quad \mathbb{J}_{ab} = \mathbb{J}_{\bar{a}\bar{b}} = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}. \quad (7.48)$$

As shown in Appendix C, subsection C.1, an arbitrary  $\mathcal{N} = 3$  superconformal transformation decomposes into two transformations in  $\mathcal{N} = 2$  superspace: (i) an  $\mathcal{N} = 2$  superconformal transformation generated by an  $\mathcal{N} = 2$  superconformal Killing vector  $\boldsymbol{\xi}$ ; (ii) an extended superconformal transformation generated by a *real* spinor superfield  $\rho^\alpha$  obeying the constraints (C.7). The action (7.44) is invariant under the  $\mathcal{N} = 2$  superconformal transformation

$$\delta\phi^a = -\boldsymbol{\xi}\phi^a - \frac{1}{2}\boldsymbol{\sigma}\phi^a. \quad (7.49)$$

It is also invariant under the extended superconformal transformation

$$\delta\phi^a = \frac{1}{2}\bar{\mathbb{D}}^2\left\{\bar{\boldsymbol{\rho}}\mathbb{J}^{ab}\frac{\partial\mathbb{K}}{\partial\phi^b}\right\}, \quad (7.50)$$

where the complex scalar  $\boldsymbol{\rho}$  and its complex conjugate  $\bar{\boldsymbol{\rho}}$  are defined as follows:

$$\rho_\alpha = \mathbb{D}_\alpha\boldsymbol{\rho} = \bar{\mathbb{D}}_\alpha\bar{\boldsymbol{\rho}}. \quad (7.51)$$

The transformation law (7.50) can be derived by applying the four-dimensional construction of [14].<sup>10</sup>

## 8 Off-shell $\mathcal{N} = 4$ superconformal sigma-models

$\mathcal{N} = 3$  supersymmetry in three dimensions is similar to and intimately related to  $\mathcal{N} = 4$ . We will show in the next section that any  $\mathcal{N} = 3$  supersymmetric sigma-model possesses a hidden  $\mathcal{N} = 4$  supersymmetry. The crucial difference between the  $\mathcal{N} = 3$  and  $\mathcal{N} = 4$  projective superspace approaches is that any  $\mathcal{N} = 3$  projective multiplet has two  $\mathcal{N} = 4$  cousins: left and right ones. This can be seen from the fact that the  $\mathcal{N} = 3$  projective superspace (7.1) turns into

$$\mathbb{M}^{3|8} \times \left\{ \mathbb{C}P_L^1 \bigcup \mathbb{C}P_R^1 \right\} = \left( \mathbb{M}^{3|8} \times \mathbb{C}P_L^1 \right) \bigcup \left( \mathbb{M}^{3|8} \times \mathbb{C}P_R^1 \right), \quad (8.1)$$

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<sup>10</sup>Strictly speaking, in order to use the 4D  $\mathcal{N} = 2$  construction of [14] for deriving eq. (7.50, we need  $\mathcal{N} = 4$  supersymmetry in three dimensions. However, it is shown in section 9 that  $\mathcal{N} = 3$  supersymmetry implies  $\mathcal{N} = 4$ .

in the  $\mathcal{N} = 4$  case, see the discussion in subsection 4.3. However, one could also consider larger supermultiplets defined on the bi-projective superspace  $\mathbb{M}^{3|8} \times \mathbb{CP}^1 \times \mathbb{CP}^1$ .

The mirror projective spaces  $\mathbb{CP}_L^1$  and  $\mathbb{CP}_R^1$  can be parametrised by homogeneous complex coordinates, or isotwistors,  $v_L = (v^i)$  and  $v_R = (v^{\bar{k}})$ , respectively. They can be used to define two maximal subsets of strictly anticommuting derivatives,  $D_\alpha^{(1)\bar{k}} := v_i D_\alpha^{i\bar{k}}$  and  $D_\alpha^{(1)i} := v_{\bar{k}} D_\alpha^{i\bar{k}}$ , such that

$$\{D_\alpha^{(1)\bar{k}}, D_\beta^{(1)\bar{l}}\} = 0 ; \quad (8.2a)$$

$$\{D_\alpha^{(1)i}, D_\beta^{(1)j}\} = 0 . \quad (8.2b)$$

These relations immediately imply that we can introduce two types of projective multiplets, left  $Q_L^{(n)}(v_L)$  and right  $Q_R^{(n)}(v_R)$  ones, which depend on the different isotwistors and obey the covariant constraints

$$D_\alpha^{(1)\bar{k}} Q_L^{(n)}(v_L) = 0 ; \quad (8.3a)$$

$$D_\alpha^{(1)i} Q_R^{(n)}(v_R) = 0 . \quad (8.3b)$$

To define  $\mathcal{N} = 4$  superconformal projective multiplets, left or right, we should again make use of the relations (7.3) and (7.4) in each of the two sectors, left and right. In particular, the  $\mathcal{N} = 4$  superconformal transformation laws of the left and right projective multiplets are respectively

$$\delta Q_L^{(n)} = -\left(\xi - \Lambda_L^{(2)} \boldsymbol{\partial}_L^{(-2)}\right) Q_L^{(n)} - n \Sigma_L Q_L^{(n)} ; \quad (8.4a)$$

$$\delta Q_R^{(n)} = -\left(\xi - \Lambda_R^{(2)} \boldsymbol{\partial}_R^{(-2)}\right) Q_R^{(n)} - n \Sigma_R Q_R^{(n)} . \quad (8.4b)$$

The objects appearing in these transformation laws have been defined in subsection 5.5. In particular, associated with the left isotwistor  $v_L = (v^i)$  is a linearly independent one  $u_L = (u_i)$  such that  $(v_L \cdot u_L) := v^i u_i \neq 0$ , and similarly in the right sector.

Next, we can introduce a  $\mathcal{N} = 4$  superconformal action principle in complete analogy with the  $\mathcal{N} = 3$  case. Consider a left real weight-2 projective multiplet  $\mathcal{L}_L^{(2)}(z, v_L)$  and a right real weight-2 projective multiplet  $\mathcal{L}_R^{(2)}(z, v_R)$ . Then, the following functional

$$\begin{aligned} S = & \frac{1}{2\pi} \oint_{\gamma_L} v_i dv^i \int d^3x D_L^{(-4)} \mathcal{L}_L^{(2)}(z, v_L) \Big|_{\theta=0} \\ & + \frac{1}{2\pi} \oint_{\gamma_R} v_{\bar{k}} dv^{\bar{k}} \int d^3x D_R^{(-4)} \mathcal{L}_R^{(2)}(z, v_R) \Big|_{\theta=0} \end{aligned} \quad (8.5)$$

is invariant under the  $\mathcal{N} = 4$  superconformal transformations. Here we have defined

$$D_L^{(-4)} := \frac{1}{48} D^{(-2)\bar{k}\bar{l}} D_{\bar{k}\bar{l}}^{(-2)} , \quad D_{\bar{k}\bar{l}}^{(-2)} := D_{\bar{k}}^{(-1)\gamma} D_{\gamma\bar{l}}^{(-1)} , \quad (8.6a)$$

$$D_R^{(-4)} := \frac{1}{48} D^{(-2)ij} D_{ij}^{(-2)} , \quad D_{ij}^{(-2)} := D_i^{(-1)\gamma} D_{\gamma j}^{(-1)} , \quad (8.6b)$$

see subsection 5.5 for the definition of spinor covariant derivatives  $D_\alpha^{(-1)i}$  and  $D_\alpha^{(-1)\bar{k}}$ .

Supersymmetric matter can be described by weight- $n$  arctic multiplets, left  $\Upsilon_L^{(n)}(v_L)$  and right  $\Upsilon_R^{(n)}(v_R)$ , and their conjugate antarctic multiplets. In order to get a better understanding of the difference between the left and the right arctic multiplets, it suffices to study the superconformal transformation properties of their component superfields<sup>11</sup>  $\Upsilon_{L,k}$  and  $\Upsilon_{R,k}$  defined by

$$\Upsilon_L^{(n)}(v_L) = (v^1)^n \Upsilon_L^{[n]}(\zeta_L) , \quad \Upsilon_L^{[n]}(\zeta_L) = \sum_{k=0}^{\infty} \Upsilon_{L,k} \zeta_L^k , \quad \zeta_L := \frac{v^2}{v^1} \quad (8.7a)$$

$$\Upsilon_R^{(n)}(v_R) = (v^{\bar{1}})^n \Upsilon_R^{[n]}(\zeta_R) , \quad \Upsilon_R^{[n]}(\zeta_R) = \sum_{k=0}^{\infty} \Upsilon_{R,k} \zeta_R^k , \quad \zeta_R := \frac{v^{\bar{2}}}{v^{\bar{1}}} \quad (8.7b)$$

considered as  $\mathcal{N} = 2$  superfields. Then, the analyticity constraints (8.3a) imply that the left components

$$\Phi_L := \Upsilon_{L,0} , \quad \Sigma_L := \Upsilon_{L,1} \quad (8.8)$$

are  $\mathcal{N} = 2$  chiral and complex linear superfields, respectively, and similarly for the right  $\mathcal{N} = 2$  superfields

$$\Phi_R := \Upsilon_{R,0} , \quad \Sigma_R := \Upsilon_{R,1} . \quad (8.9)$$

The other component  $\mathcal{N} = 2$  superfields are complex unconstrained.

We know from Appendix C, subsection C.2, that any arbitrary  $\mathcal{N} = 4$  superconformal transformation decomposes into three transformations in  $\mathcal{N} = 2$  superspace: (i) an  $\mathcal{N} = 2$  superconformal transformation generated by an  $\mathcal{N} = 2$  superconformal Killing vector  $\xi$ ; (ii) an extended superconformal transformation generated by a *complex* spinor superfield  $\rho^\alpha$  obeying the constraints (C.11); (iii) a shadow U(1) rotation. Using the  $\mathcal{N} = 4$  superconformal transformation laws (8.4a) and (8.4b), it is a simple exercise to work out the transformations of the  $\mathcal{N} = 2$  component superfields.

The left and the right arctic multiplets turn out to have identical  $\mathcal{N} = 2$  superconformal transformations, so we will omit the label ‘L’ and ‘R.’ Specifically, one can show that the component superfields  $\Upsilon_k$  transform as

$$\delta_\xi \Upsilon_k = - \left( \xi + \frac{n-k}{2} \sigma + \frac{k}{2} \bar{\sigma} \right) \Upsilon_k . \quad (8.10)$$

---

<sup>11</sup>For each of the mirror spaces  $\mathbb{CP}_L^1$  and  $\mathbb{CP}_R^1$ , we choose its north chart to define the expansions (8.7a) and (8.7b).

As already mentioned, the leading components  $\Phi := \Upsilon_0$  and  $\Sigma := \Upsilon_1$  are chiral and complex linear respectively. The transformation law (8.10) implies

$$\delta_{\boldsymbol{\xi}} \Phi = -\left(\boldsymbol{\xi} + \frac{n}{2}\boldsymbol{\sigma}\right)\Phi \longrightarrow \bar{\mathbb{D}}_{\alpha}\delta_{\boldsymbol{\xi}}\Phi = 0 \quad (8.11a)$$

$$\delta_{\boldsymbol{\xi}} \Sigma = -\left(\boldsymbol{\xi} + \frac{n-1}{2}\boldsymbol{\sigma} + \frac{1}{2}\bar{\boldsymbol{\sigma}}\right)\Sigma \longrightarrow \bar{\mathbb{D}}^2\delta_{\boldsymbol{\xi}}\Sigma = 0 . \quad (8.11b)$$

In the case of a weight-one multiplet,  $n = 1$ , the  $\boldsymbol{\sigma}$ -term in the variation  $\delta_{\boldsymbol{\xi}}\Sigma$  drops out.

The left and the right arctic multiplets transform almost identically under the shadow  $U(1)$  rotation:

$$\delta_{\alpha}\Upsilon_{L,k} = -\frac{(n-2k)}{2}i\alpha\Upsilon_{L,k} , \quad \delta_{\alpha}\Upsilon_{R,k} = \frac{(n-2k)}{2}i\alpha\Upsilon_{R,k} . \quad (8.12)$$

A real difference between the left and the right arctic multiplets occurs only in the sector of extended superconformal transformations. The left weight- $n$  arctic multiplet can be shown to transform as follows:

$$\delta\Upsilon_{L,0} = \left(\rho^{\alpha}\bar{\mathbb{D}}_{\alpha} - \frac{1}{2}(\bar{\mathbb{D}}_{\alpha}\rho^{\alpha})\right)\Upsilon_{L,1} , \quad (8.13a)$$

$$\begin{aligned} \delta\Upsilon_{L,k} = & \left(\bar{\rho}^{\alpha}\mathbb{D}_{\alpha} + \frac{k-n-1}{2}(\mathbb{D}_{\alpha}\bar{\rho}^{\alpha})\right)\Upsilon_{L,k-1} \\ & + \left(\rho^{\alpha}\bar{\mathbb{D}}_{\alpha} - \frac{k+1}{2}(\bar{\mathbb{D}}_{\alpha}\rho^{\alpha})\right)\Upsilon_{L,k+1} , \quad k > 0 . \end{aligned} \quad (8.13b)$$

For the right weight- $n$  arctic multiplet we get

$$\delta\Upsilon_{R,0} = -\left(\bar{\rho}^{\alpha}\bar{\mathbb{D}}_{\alpha} - \frac{1}{2}(\bar{\mathbb{D}}_{\alpha}\bar{\rho}^{\alpha})\right)\Upsilon_{R,1} , \quad (8.14a)$$

$$\begin{aligned} \delta\Upsilon_{R,k} = & -\left(\rho^{\alpha}\mathbb{D}_{\alpha} + \frac{k-n-1}{2}(\mathbb{D}_{\alpha}\rho^{\alpha})\right)\Upsilon_{R,k-1} \\ & -\left(\bar{\rho}^{\alpha}\bar{\mathbb{D}}_{\alpha} - \frac{k+1}{2}(\bar{\mathbb{D}}_{\alpha}\bar{\rho}^{\alpha})\right)\Upsilon_{R,k+1} , \quad k > 0 . \end{aligned} \quad (8.14b)$$

We see that the transformation law of the right superfields,  $\Upsilon_{R,k}$ , can be obtained from that of the left ones,  $\Upsilon_{L,k}$ , by applying the replacement:  $\Upsilon_{L,k} \rightarrow \Upsilon_{R,k}$  and  $\rho_{\alpha} \rightarrow -\bar{\rho}_{\alpha}$ .

The leading left  $\mathcal{N} = 2$  superfields (8.8) are chiral and complex linear, respectively, and similarly for the right  $\mathcal{N} = 2$  superfields (8.9). It is important to check that their variations under the extended superconformal transformation obey the same constraints. To make manifest the chirality of  $\delta\Phi_L$  and  $\delta\Phi_R$ , we point out the following. It follows from the constraints (C.11) that the superconformal parameters  $\rho_{\alpha}$  and  $\bar{\rho}_{\alpha}$  can be represented in the form:

$$\rho_{\alpha} = \bar{\mathbb{D}}_{\alpha}\bar{\boldsymbol{\rho}}_L , \quad \bar{\rho}_{\alpha} = \mathbb{D}_{\alpha}\boldsymbol{\rho}_L ; \quad (8.15a)$$

$$\rho_{\alpha} = \mathbb{D}_{\alpha}\boldsymbol{\rho}_R , \quad \bar{\rho}_{\alpha} = \bar{\mathbb{D}}_{\alpha}\bar{\boldsymbol{\rho}}_R , \quad (8.15b)$$

for some complex scalars  $\rho_L$  and  $\rho_R$ . Now, the variations (8.13a) and (8.14a) can be rewritten in the form:

$$\delta\Phi_L = -\frac{1}{2}\bar{\mathbb{D}}^2(\bar{\rho}_L\Sigma_L) , \quad (8.16a)$$

$$\delta\Phi_R = -\frac{1}{2}\bar{\mathbb{D}}^2(\bar{\rho}_R\Sigma_R) , \quad (8.16b)$$

with  $\bar{\mathbb{D}}^2 := \bar{\mathbb{D}}_\alpha\bar{\mathbb{D}}^\alpha$ . These expressions show that  $\delta\Phi_L$  and  $\delta\Phi_R$  are indeed chiral. Let us consider eq. (8.13b) with  $k = 1$  in order to check that  $\delta\Sigma_L$  satisfies the condition  $\bar{\mathbb{D}}^2\delta\Sigma_L = 0$ . We have

$$\begin{aligned} \delta\Sigma_L &= \left(\bar{\rho}^\alpha\mathbb{D}_\alpha - \frac{n}{2}(\mathbb{D}_\alpha\bar{\rho}^\alpha)\right)\Phi_L + \left(\rho^\alpha\bar{\mathbb{D}}_\alpha - (\bar{\mathbb{D}}_\alpha\rho^\alpha)\right)\Upsilon_{L,2} \\ &= \left(\bar{\rho}^\alpha\mathbb{D}_\alpha - \frac{n}{2}(\mathbb{D}_\alpha\bar{\rho}^\alpha)\right)\Phi_L - \bar{\mathbb{D}}_\alpha\left(\rho^\alpha\Upsilon_{L,2}\right) . \end{aligned} \quad (8.17)$$

Here the second term in the second line is clearly complex linear. To prove that the first term is also complex linear, it suffices to use the constraints (C.11) as well as the chirality of  $\Phi_L$ .

In conclusion, we write down a general  $\mathcal{N} = 4$  superconformal sigma-model described by left and right weight-one arctic multiplets and their conjugates

$$\begin{aligned} S &= \frac{1}{2\pi i} \oint_{\gamma_L} \frac{d\zeta_L}{\zeta_L} \int d^3x d^2\theta d^2\bar{\theta} K_L(\Upsilon_L, \check{\Upsilon}_L) \\ &\quad + \frac{1}{2\pi i} \oint_{\gamma_R} \frac{d\zeta_R}{\zeta_R} \int d^3x d^2\theta d^2\bar{\theta} K_R(\Upsilon_R, \check{\Upsilon}_R) . \end{aligned} \quad (8.18)$$

Here the Kähler potentials  $K_L$  and  $K_R$  obey homogeneity conditions of the type (7.31).

## 9 $\mathcal{N} = 3$ SUSY implies $\mathcal{N} = 4$ SUSY

It is the accepted lore that  $\mathcal{N} = 3$  supersymmetry in three space-time dimensions implies  $\mathcal{N} = 4$  supersymmetry.<sup>12</sup> Here we provide two proofs of this claim by considering nonlinear sigma-models that possess different amounts of manifestly realised supersymmetry: (i)  $\mathcal{N} = 2$ ; and (ii)  $\mathcal{N} = 3$ .

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<sup>12</sup>The standard argument (see, e.g., [65]) is as follows:  $\mathcal{N}$ -extended supersymmetry requires  $\mathcal{N} - 1$  anti-commuting complex structures. In the case  $\mathcal{N} = 3$ , the target space has two such structures,  $I$  and  $J$ . Their product  $K := IJ$  is a third complex structure which anticommutes with  $I$  and  $J$ , and therefore the sigma-model is  $\mathcal{N} = 4$  supersymmetric. This argument tells us nothing about off-shell supersymmetry.

## 9.1 Analysis in $\mathcal{N} = 2$ superspace

We start from a general 3D  $\mathcal{N} = 2$  supersymmetric nonlinear  $\sigma$ -model [68]

$$S = \int d^3x d^2\theta d^2\bar{\theta} K(\phi^a, \bar{\phi}^{\bar{a}}) , \quad \bar{\mathbb{D}}_{\beta}\phi^a = 0 , \quad (9.1)$$

with  $K(\phi, \bar{\phi})$  the Kähler potential of a Kähler manifold  $\mathcal{M}$ , and look for those restrictions on the target space geometry which guarantee the existence of a hidden supersymmetry. We emphasise that the sigma-model under consideration is not required to be superconformal.

The first argument why  $\mathcal{N} = 3$  implies  $\mathcal{N} = 4$  is based on an explicit calculation. Building on the four-dimensional analysis [66, 67], a general ansatz for an additional supersymmetry is of the form

$$\delta\phi^a = \frac{1}{2}\bar{\mathbb{D}}^2(\bar{\rho}\bar{\Omega}^a) , \quad \delta\bar{\phi}^{\bar{a}} = \frac{1}{2}\mathbb{D}^2(\rho\Omega^{\bar{a}}) , \quad (9.2)$$

where  $\Omega^a(\phi, \bar{\phi})$  is a function associated with the Kähler manifold  $\mathcal{M}$ , and  $\bar{\rho}$  is a constant antichiral *superfield* satisfying

$$\mathbb{D}_{\alpha}\bar{\rho} = \partial_{\alpha\beta}\bar{\rho} = \bar{\mathbb{D}}^2\bar{\rho} = 0 . \quad (9.3)$$

In three dimensions it is furthermore possible to choose the spinor component of  $\rho$  to be real up to an arbitrary constant phase  $\lambda$

$$e^{-i\lambda}\mathbb{D}_{\alpha}\rho| = e^{i\lambda}\bar{\mathbb{D}}_{\alpha}\bar{\rho}| , \quad (9.4)$$

with  $\lambda \in \mathbb{R}$  some fixed parameter for the transformation under consideration. A real spinor parameter corresponds to one additional (third) supersymmetry. Thus  $\bar{\rho}$  has the following explicit form:

$$\bar{\rho} = \bar{\tau} + e^{-i\lambda}\epsilon_{\alpha}\bar{\theta}^{\alpha} , \quad \bar{\tau} = \text{const} , \quad \epsilon_{\alpha} = \bar{\epsilon}_{\alpha} = \text{const} . \quad (9.5)$$

In fact, the third supersymmetry transformation is obtained by setting  $\bar{\tau} = 0$ . However, commuting the manifestly realised first and second supersymmetry transformations with the third one results in a *central charge transformation* which corresponds to the choice  $\bar{\rho} = \bar{\tau} \in \mathbb{C}$  in (9.2). For our proof below, it is not necessary to assume  $\bar{\tau}$  to be complex.

We would like to show that there is no *special* choice of  $\lambda$  in (9.5) for which the supersymmetry variation cancels between the  $\rho$  terms and the  $\bar{\rho}$  terms. Varying the



action (9.1) gives

$$\begin{aligned}\delta_{\rho,\bar{\rho}}S &= \frac{1}{2} \int d^3x d^2\theta d^2\bar{\theta} K_a \bar{\mathbb{D}}^2(\bar{\rho}\bar{\Omega}^a) + \text{c.c.} \\ &= -\frac{1}{2} \int d^3x d^2\theta d^2\bar{\theta} g_{a\bar{b}}(\bar{\mathbb{D}}_\alpha\bar{\phi}^{\bar{b}})\bar{\mathbb{D}}^\alpha(\bar{\rho}\bar{\Omega}^a) + \text{c.c.} ,\end{aligned}\quad (9.6)$$

with  $g_{a\bar{b}} = K_{a\bar{b}} := \partial_a\partial_{\bar{b}}K$  the Kähler metric. Choosing here  $\bar{\rho} = \bar{\tau} = \text{const}$ , the variation of the action reduces to

$$\delta_{\tau,\bar{\tau}}S = -\frac{1}{2}\bar{\tau} \int d^3x d^2\theta d^2\bar{\theta} \bar{\omega}_{\bar{b}\bar{c}}(\bar{\mathbb{D}}_\alpha\bar{\phi}^{\bar{b}})\bar{\mathbb{D}}^\alpha\bar{\phi}^{\bar{c}} + \text{c.c.} , \quad (9.7)$$

where we have denoted

$$\bar{\omega}_{\bar{b}\bar{c}} := g_{\bar{b}a} \bar{\Omega}^a_{,\bar{c}} , \quad \bar{\Omega}^a_{,\bar{c}} := \partial_{\bar{c}}\bar{\Omega}^a . \quad (9.8)$$

The two terms in (9.7) can be seen to have different functional form. The condition  $\delta_{\tau,\bar{\tau}}S = 0$  is therefore equivalent to the requirement that each term on the right of (9.7) vanishes separately, and thus

$$\int d^3x d^2\theta d^2\bar{\theta} \bar{\omega}_{\bar{b}\bar{c}}(\bar{\mathbb{D}}_\alpha\bar{\phi}^{\bar{b}})\bar{\mathbb{D}}^\alpha\bar{\phi}^{\bar{c}} = 0 . \quad (9.9)$$

This holds if

$$\bar{\omega}_{\bar{b}\bar{c}} = -\bar{\omega}_{\bar{c}\bar{b}} , \quad (9.10)$$

and hence the target space is endowed with the *two-form*<sup>13</sup>  $\omega_{bc} = g_{b\bar{a}} \Omega^{\bar{a}}_{,c}$  and its conjugate  $\bar{\omega}_{\bar{b}\bar{c}}$ . We conclude that the action is invariant under the central charge transformation when  $\omega$  is antisymmetric. Thus there is no way that the  $\tau$  and  $\bar{\tau}$  variations could possibly cancel each other without the variations being separately zero.

Now, the variation of the action, eq. (9.6), turns into

$$\delta_{\rho,\bar{\rho}}S = -\frac{1}{2}e^{-i\lambda}\epsilon_\alpha \int d^3x d^2\theta d^2\bar{\theta} g_{a\bar{b}} \bar{\Omega}^a \bar{\mathbb{D}}_\alpha\bar{\phi}^{\bar{b}} + \text{c.c.} \quad (9.11)$$

We see that the two terms in  $\delta_{\rho,\bar{\rho}}S$  have different functional form. Imposing the condition  $\delta_{\rho,\bar{\rho}}S = 0$  is equivalent to the fact that each term should vanish separately. Therefore, we have to require

$$\begin{aligned}\delta_{\bar{\rho}}S &= \frac{1}{2}e^{-i\lambda}\epsilon^\alpha \int d^3x d^2\theta d^2\bar{\theta} g_{a\bar{b}} \bar{\Omega}^a \bar{\mathbb{D}}_\alpha\bar{\phi}^{\bar{b}} \\ &= \frac{1}{2}e^{-i\lambda}\epsilon^\alpha \int d^3x d^2\theta d^2\bar{\theta} \bar{\Omega}^a \bar{\mathbb{D}}_\alpha K_a \equiv 0\end{aligned}\quad (9.12)$$

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<sup>13</sup>On the mass shell, the supersymmetry transformation (9.2) takes the form:  $\delta\phi^a = e^{-i\lambda}\epsilon_\alpha \bar{\Omega}^a_{,\bar{b}} \bar{\mathbb{D}}^\alpha\bar{\phi}^{\bar{b}}$ . Since  $\delta\phi^a$  should be a vector field on  $\mathcal{M}$ , we conclude that  $\bar{\Omega}^a_{,\bar{b}}$  is a tensor field on  $\mathcal{M}$ , and therefore  $\omega_{ab}$  is a two-form.

for the action to be invariant under the third supersymmetry.

Before starting to analyse the condition (9.12), we wish to make an important observation. Let us forget for a moment that  $\bar{\omega}_{\bar{b}\bar{c}}$  defined by (9.8) is antisymmetric, eq. (9.10). Then, we can derive several identities:

$$\begin{aligned}
B_{\bar{b}\bar{c}\bar{d}} &:= K_{a\bar{b}\bar{c}}\bar{\Omega}^a_{\cdot\bar{d}} + K_{a\bar{d}}\bar{\Omega}^a_{\cdot\bar{b}\bar{c}} \\
&= \nabla_{\bar{c}}\bar{\omega}_{\bar{d}\bar{b}} + \Gamma_{\bar{b}\bar{c}}^{\bar{e}}(\bar{\omega}_{\bar{d}\bar{e}} + \bar{\omega}_{\bar{e}\bar{d}}) \\
&= \frac{1}{2}\left(\partial_{\bar{b}}(\bar{\omega}_{\bar{c}\bar{d}} + \bar{\omega}_{\bar{d}\bar{c}}) + \partial_{\bar{c}}(\bar{\omega}_{\bar{b}\bar{d}} + \bar{\omega}_{\bar{d}\bar{b}}) - \partial_{\bar{d}}(\bar{\omega}_{\bar{b}\bar{c}} + \bar{\omega}_{\bar{c}\bar{b}})\right). \tag{9.13}
\end{aligned}$$

If eq. (9.10) holds, then the relation (9.13) leads to the important result:

$$\nabla_{\bar{a}}\bar{\omega}_{\bar{b}\bar{c}} = 0. \tag{9.14}$$

A non-trivial piece of information can be extracted from the condition (9.12) without doing hard calculations (compare with [14]). Eq. (9.12) tells us that the functional in the second line must vanish identically. Let us vary this functional with respect to the antichiral superfields  $\bar{\phi}$ s while keeping the chiral ones  $\phi$ s fixed. Since  $\bar{\omega}_{\bar{b}\bar{c}}$  is antisymmetric, we obtain

$$\delta_{\bar{\phi}} \int d^3x d^2\theta d^2\bar{\theta} \bar{\Omega}^a \bar{\mathbb{D}}_{\alpha} K_a = -2 \int d^3x d^2\theta d^2\bar{\theta} \bar{\omega}_{\bar{b}\bar{c}} \delta\bar{\phi}^{\bar{b}} \bar{\mathbb{D}}_{\alpha} \bar{\phi}^{\bar{c}}. \tag{9.15}$$

To make this variation vanish, one has to impose the condition

$$\nabla_a \bar{\omega}_{\bar{b}\bar{c}} = \partial_a \bar{\omega}_{\bar{b}\bar{c}} = 0, \tag{9.16}$$

which means that  $\bar{\omega}_{\bar{b}\bar{c}}$  is anti-holomorphic,  $\bar{\omega}_{\bar{b}\bar{c}} = \bar{\omega}_{\bar{b}\bar{c}}(\bar{\phi})$ . Indeed, since  $\bar{\phi}$  and  $\delta\bar{\phi}$  are antichiral, we then get

$$\int d^3x d^2\theta d^2\bar{\theta} \bar{\omega}_{\bar{b}\bar{c}} \delta\bar{\phi}^{\bar{b}} \bar{\mathbb{D}}_{\alpha} \bar{\phi}^{\bar{c}} = -\frac{1}{4} \int d^3x d^2\bar{\theta} \bar{\omega}_{\bar{b}\bar{c}} \delta\bar{\phi}^{\bar{b}} \bar{\mathbb{D}}^2 \bar{\mathbb{D}}_{\alpha} \bar{\phi}^{\bar{c}} = 0.$$

The conditions (9.10), (9.14) and (9.16) prove to be sufficient for demonstrate that eq. (9.12) indeed holds. It is in fact instructive to compute  $\delta_{\bar{\rho}} S$  directly without making use of (9.10), (9.14) and (9.16).

Tedious calculations lead to

$$\begin{aligned}
\delta_{\bar{\rho}} S &= \frac{1}{16} e^{-i\lambda} \epsilon_{\alpha} \int d^3x \mathbb{D}^2 \left\{ \mathbb{D}^2 K_a \bar{\mathbb{D}}^{\alpha} \bar{\Omega}^a \right\} \\
&= \frac{e^{-i\lambda} \epsilon^{\alpha}}{16} \int d^3x \left\{ 8\bar{\omega}_{\bar{b}\bar{c}} \left( \square \bar{\phi}^{\bar{b}} \bar{\mathbb{D}}_{\alpha} \bar{\phi}^{\bar{c}} + \square \bar{\phi}^{\bar{c}} \bar{\mathbb{D}}_{\alpha} \bar{\phi}^{\bar{b}} \right) + 16B_{\bar{b}\bar{c}\bar{d}} \partial^{\beta\gamma} \bar{\phi}^{\bar{b}} \bar{\mathbb{D}}_{\gamma} \bar{\phi}^{\bar{d}} \partial_{\alpha\beta} \bar{\phi}^{\bar{c}} \right. \\
&\quad + \nabla_a \bar{\omega}_{\bar{b}\bar{c}} \left( 16\partial^{\beta\gamma} \phi^a \bar{\mathbb{D}}_{\gamma} \bar{\phi}^{\bar{b}} \partial_{\alpha\beta} \bar{\phi}^{\bar{c}} - \mathbb{D}^2 \phi^a \bar{\mathbb{D}}^2 \bar{\phi}^{\bar{b}} \bar{\mathbb{D}}_{\alpha} \bar{\phi}^{\bar{c}} \right. \\
&\quad + 8i \mathbb{D}^{\beta} \phi^a \partial_{\beta\gamma} \bar{\mathbb{D}}_{\gamma} \bar{\phi}^{\bar{b}} \bar{\mathbb{D}}_{\alpha} \bar{\phi}^{\bar{c}} + 4i \mathbb{D}^{\beta} \phi^a \bar{\mathbb{D}}^2 \bar{\phi}^{\bar{b}} \partial_{\alpha\beta} \bar{\phi}^{\bar{c}} \left. \right) - \partial_a \nabla_b \bar{\omega}_{\bar{c}\bar{d}} \mathbb{D}^{\beta} \phi^a \mathbb{D}_{\beta} \phi^b \bar{\mathbb{D}}^2 \bar{\phi}^{\bar{c}} \bar{\mathbb{D}}_{\alpha} \bar{\phi}^{\bar{d}} \\
&\quad + \partial_a (\Gamma_{\bar{b}\bar{c}}^{\bar{e}} \bar{\omega}_{\bar{e}\bar{d}}) \left( - \mathbb{D}^2 \phi^a \bar{\mathbb{D}}_{\beta} \bar{\phi}^{\bar{b}} \bar{\mathbb{D}}^{\beta} \bar{\phi}^{\bar{c}} \bar{\mathbb{D}}_{\alpha} \bar{\phi}^{\bar{d}} \right. \\
&\quad + 4i \mathbb{D}_{\gamma} \phi^a \partial^{\gamma\beta} \bar{\phi}^{\bar{b}} \bar{\mathbb{D}}_{\beta} \bar{\phi}^{\bar{c}} \bar{\mathbb{D}}_{\alpha} \bar{\phi}^{\bar{d}} + 4i \mathbb{D}_{\gamma} \phi^a \partial^{\gamma\beta} \bar{\phi}^{\bar{c}} \bar{\mathbb{D}}_{\beta} \bar{\phi}^{\bar{b}} \bar{\mathbb{D}}_{\alpha} \bar{\phi}^{\bar{d}} + 4i \mathbb{D}^{\gamma} \phi^a \bar{\mathbb{D}}_{\beta} \bar{\phi}^{\bar{b}} \bar{\mathbb{D}}^{\beta} \bar{\phi}^{\bar{c}} \partial_{\gamma\alpha} \bar{\phi}^{\bar{d}} \left. \right) \\
&\quad \left. + \partial_a \partial_b (\Gamma_{\bar{c}\bar{d}}^{\bar{f}} \bar{\omega}_{\bar{f}\bar{e}}) \mathbb{D}^{\gamma} \phi^a \mathbb{D}_{\gamma} \phi^b \bar{\mathbb{D}}^{\beta} \bar{\phi}^{\bar{c}} \bar{\mathbb{D}}_{\beta} \bar{\phi}^{\bar{d}} \bar{\mathbb{D}}_{\alpha} \bar{\phi}^{\bar{e}} \right\}. \tag{9.17}
\end{aligned}$$

We see that the contributions in the first, second and third lines vanish due to the conditions (9.10), (9.14) and (9.16). The remaining contributions in (9.17) are proportional either to

$$\partial_a (\Gamma_{\bar{b}\bar{c}}^{\bar{e}} \bar{\omega}_{\bar{e}\bar{d}}) = (\partial_a \Gamma_{\bar{b}\bar{c}}^{\bar{e}}) \bar{\omega}_{\bar{e}\bar{d}} = R_{\bar{b}a\bar{c}}^{\bar{e}} \bar{\omega}_{\bar{e}\bar{d}} \tag{9.18}$$

or to its derivative,  $\partial_a \partial_b (\Gamma_{\bar{c}\bar{d}}^{\bar{f}} \bar{\omega}_{\bar{f}\bar{e}})$ . Now the fact that  $\bar{\omega}_{\bar{b}\bar{c}}$  is covariantly constant, implies that  $R_{\bar{b}a\bar{c}}^{\bar{e}} \bar{\omega}_{\bar{e}\bar{d}}$  is symmetric in all barred indices which is enough for all the remaining terms to vanish.

We have demonstrated that the conditions (9.10), (9.14) and (9.16) guarantee that the sigma-model action (9.1) is invariant under the additional third supersymmetry transformation (9.2), with  $\bar{\rho}$  given by eq. (9.5). The important point is that  $\bar{\rho}$  in (9.2) depends on the phase factor  $e^{-i\lambda}$ , where  $\lambda$  is a fixed parameter characterising the third supersymmetry transformation. However, since  $\lambda$  does not show up in the conditions (9.10), (9.14) and (9.16), the action (9.1) is invariant under the additional third supersymmetry transformation (9.2) and (9.5) in which  $\lambda$  is *completely arbitrary*. This means that the action (9.1) is invariant under hidden supersymmetry transformations of the form:

$$\delta \phi^a = \frac{1}{2} \bar{\mathbb{D}}^2 \left( \bar{\rho}(\bar{\theta}) \bar{\Omega}^a(\phi, \bar{\phi}) \right), \quad \bar{\rho}(\bar{\theta}) = \bar{\tau} + \bar{\epsilon}_{\alpha} \bar{\theta}^{\alpha}, \tag{9.19}$$

with arbitrary *complex* constant parameters  $\bar{\tau}$  and  $\bar{\epsilon}_{\alpha}$ . Therefore  $\mathcal{N} = 3$  supersymmetry implies  $\mathcal{N} = 4$  supersymmetry.

We now turn to describing the second argument why  $\mathcal{N} = 3$  supersymmetry implies  $\mathcal{N} = 4$ . For this we rewrite the third supersymmetry transformation (9.2) as

$$\delta_\lambda \phi^a = \frac{1}{2} \bar{\mathbb{D}}^2 \left( \bar{\rho}_\lambda(\bar{\theta}) \bar{\Omega}^a(\phi, \bar{\phi}) \right) . \quad (9.20)$$

The sigma-model action (9.1) is clearly  $R$ -invariant. It does not change under  $U(1)$  transformations

$$\phi^a(\theta) \rightarrow \phi'^a(\theta) = \phi^a(e^{i\psi}\theta) , \quad \bar{\phi}^{\bar{b}}(\bar{\theta}) \rightarrow \bar{\phi}'^{\bar{b}}(\bar{\theta}) = \bar{\phi}^{\bar{b}}(e^{-i\psi}\bar{\theta}) , \quad \psi \in \mathbb{R} . \quad (9.21)$$

Commuting such an infinitesimal transformation with the supersymmetry one, eq. (9.20), results in a new supersymmetry transformation of the form:

$$\delta_{\lambda+\delta\lambda} \phi^a = \frac{1}{2} \bar{\mathbb{D}}^2 \left( \bar{\rho}_{\lambda+\delta\lambda}(\bar{\theta}) \bar{\Omega}^a(\phi, \bar{\phi}) \right) , \quad (9.22)$$

with  $\delta\lambda \neq 0$ . Therefore, if the action (9.1) is invariant under the third supersymmetry (9.20) and (9.5), for some fixed  $\lambda$ , it is in fact invariant under a one-parameter family of supersymmetry transformations corresponding to all possible values for  $\lambda \in \mathbb{R}$  in (9.20). This means that  $\mathcal{N} = 3$  supersymmetry implies  $\mathcal{N} = 4$  supersymmetry. Instead of the ansatz (9.20), we can now look for a hidden supersymmetry transformation of the form (9.19). Therefore, we can recycle, word for word, the four-dimensional derivation given in [14] of the results of [67] devoted to the formulation of general 4D  $\mathcal{N} = 2$  supersymmetric nonlinear sigma-models in terms of  $\mathcal{N} = 1$  chiral superfields.

On the mass shell,

$$\bar{\mathbb{D}}^2 K_a = 0 , \quad (9.23)$$

and the first and the second supersymmetry transformations generate the  $\mathcal{N} = 2$  super-Poincaré algebra *without* central charge provided

$$\bar{\Omega}^a_{,\bar{c}} \Omega^{\bar{c}}_{,b} = -\delta^a_b . \quad (9.24)$$

In fact, the closure of the supersymmetry algebra requires two more conditions

$$\bar{\mathbb{D}}^2 \bar{\Omega}^a = 0 , \quad (9.25)$$

$$\bar{\Omega}^d_{,\bar{b}} \nabla_d \bar{\Omega}^a_{,\bar{c}} - \bar{\Omega}^d_{,\bar{c}} \nabla_d \bar{\Omega}^a_{,\bar{b}} = 0 . \quad (9.26)$$

They hold due to (9.16) – (9.24). The detailed derivation of the above results can be found in [14].

Let  $J \equiv J_3$  be the complex structure chosen on the target space  $\mathcal{M}$ ,

$$J_3 = \begin{pmatrix} i\delta^a_b & 0 \\ 0 & -i\delta^{\bar{a}}_{\bar{b}} \end{pmatrix} . \quad (9.27)$$

The above consideration shows that there are two more complex structures defined as

$$J_1 = \begin{pmatrix} 0 & \bar{\Omega}^a_{\bar{b}} \\ \Omega^{\bar{a}}_{a,b} & 0 \end{pmatrix} , \quad J_2 = \begin{pmatrix} 0 & i\bar{\Omega}^a_{\bar{b}} \\ -i\Omega^{\bar{a}}_{a,b} & 0 \end{pmatrix} \quad (9.28)$$

such that  $\mathcal{M}$  is Kähler with respect to all of them, and the operators  $J_A = (J_1, J_2, J_3)$  form the quaternionic algebra:

$$J_A J_B = -\delta_{AB} \mathbb{1} + \varepsilon_{ABC} J_C . \quad (9.29)$$

As a result, it has been demonstrated that the target space  $\mathcal{M}$  is a hyperkähler manifold.

As is seen from (9.28), the complex structures are given in terms of the tensor fields  $\bar{\Omega}^a_{\bar{b}}$  and  $\Omega^{\bar{a}}_{a,b}$ , while the supersymmetry transformation (9.20) involves  $\bar{\Omega}^a$  and  $\Omega^{\bar{a}}$ . The latter can be constructed using the Kähler potential [67]:

$$\bar{\Omega}^a = \omega^{ab}(\phi) K_b(\phi, \bar{\phi}) . \quad (9.30)$$

Under the Kähler transformations

$$K(\phi, \bar{\phi}) \longrightarrow K(\phi, \bar{\phi}) + \Lambda(\phi) + \bar{\Lambda}(\bar{\phi}) , \quad (9.31)$$

$\bar{\Omega}^a$  changes as follows:  $\omega^{ab} K_b \rightarrow \omega^{ab} K_b + \omega^{ab} \Lambda_b$ . However, the supersymmetry variation  $\delta\phi^a = \frac{1}{2}\bar{\mathbb{D}}^2(\bar{\rho}(\bar{\theta}) \bar{\Omega}^a)$  in (9.20) is invariant under the Kähler transformations.

## 9.2 Off-shell $\mathcal{N} = 3$ SUSY implies off-shell $\mathcal{N} = 4$ SUSY

Consider the off-shell  $\mathcal{N} = 3$  superconformal sigma-model generated by the Lagrangian (7.30) and (7.31). Upon reduction to  $\mathcal{N} = 2$  superspace, its action takes the form (7.34). Comparing this action with that of the off-shell  $\mathcal{N} = 4$  superconformal sigma-model formulated in  $\mathcal{N} = 2$  superspace, eq. (8.18), we see that the former is identical in its form with either the left or the right sector of the latter. This means that the theory (7.34) can be lifted to an off-shell  $\mathcal{N} = 4$  superconformal sigma-model realised only in terms of left weight-one arctic multiplets and their conjugates, or only in terms of their right mirrors. If the Kählerian cone  $\mathcal{M}$ , for which  $K(\Phi, \bar{\Phi})$  is the Kähler potential, factorises,  $\mathcal{M} = \mathcal{M}_1 \times \mathcal{M}_2$ , then we can use both left and right arctic multiplets and their conjugates in order to realise  $\mathcal{N} = 4$  extensions of the two sectors of the  $\mathcal{N} = 3$  sigma-model.

## 10 General $\mathcal{N} = 3, 4$ superconformal sigma-models

We now have all prerequisites available to develop a chiral formulation in  $\mathcal{N} = 2$  superspace for the most general  $\mathcal{N} = 3$  and  $\mathcal{N} = 4$  superconformal nonlinear  $\sigma$ -models. Given a hyperkähler cone  $\mathcal{M}$ , we pick one of its complex structures, say  $J_3$ , and introduce complex coordinates  $\phi^a$  compatible with it. In these coordinates,  $J_3$  has the form (9.27). Two other complex structures,  $J_1$  and  $J_2$ , become

$$J_1 = \begin{pmatrix} 0 & g^{a\bar{c}}\bar{\omega}_{\bar{c}b} \\ g^{\bar{a}c}\omega_{cb} & 0 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & i g^{a\bar{c}}\bar{\omega}_{\bar{c}b} \\ -i g^{\bar{a}c}\omega_{cb} & 0 \end{pmatrix}, \quad (10.1)$$

where  $g_{a\bar{b}}$  be the hyperkähler metric, and  $\omega_{ab}$  the holomorphic symplectic two-form. Let  $\chi = (\chi^a(\phi), \bar{\chi}^{\bar{b}}(\bar{\phi}))$  be the homothetic conformal Killing vector field associated with  $\mathcal{M}$ ,

$$\nabla_a \chi^b = \delta_a^b, \quad \bar{\nabla}_{\bar{a}} \chi^b = \bar{\partial}_{\bar{a}} \chi^b = 0 \quad (10.2a)$$

$$\chi_a := g_{a\bar{b}} \bar{\chi}^{\bar{b}} = \partial_a K, \quad g_{a\bar{b}} = \partial_a \bar{\partial}_{\bar{b}} K, \quad (10.2b)$$

with the hyperkähler potential  $K$  given by

$$K = g_{a\bar{b}} \chi^a \bar{\chi}^{\bar{b}} = K_a \chi^a. \quad (10.3)$$

With this hyperkähler potential chosen, our goal is to prove that the  $\mathcal{N} = 4$  supersymmetric  $\sigma$ -model

$$S = \int d^3x d^2\theta d^2\bar{\theta} K(\phi^a, \bar{\phi}^{\bar{b}}), \quad \bar{\mathbb{D}}_{\beta} \phi^a = 0 \quad (10.4)$$

is  $\mathcal{N} = 4$  superconformal. In accordance with the analysis given in subsection 6.2, the action (10.4) is invariant under the  $\mathcal{N} = 2$  superconformal transformation

$$\delta \phi^a = -\xi \phi^a - \frac{1}{2} \sigma \chi^a(\phi), \quad (10.5)$$

with  $\xi$  an arbitrary  $\mathcal{N} = 2$  superconformal Killing vector.

### 10.1 $\mathcal{N} = 3$ superconformal invariance

To start with, we prove that the action (10.4) is  $\mathcal{N} = 3$  superconformal. As shown in Appendix C, subsection C.1, an arbitrary  $\mathcal{N} = 3$  superconformal transformation decomposes into two transformations in  $\mathcal{N} = 2$  superspace: (i) an  $\mathcal{N} = 2$  superconformal transformation generated by an  $\mathcal{N} = 2$  superconformal Killing vector  $\xi$ ; (ii) an extended

superconformal transformation generated by a *real* spinor superfield  $\rho^\alpha(x, \theta, \bar{\theta})$  obeying the constraints

$$\mathbb{D}_{(\alpha}\rho_{\beta)} = \bar{\mathbb{D}}_{(\alpha}\rho_{\beta)} = 0 , \quad (10.6)$$

and therefore

$$\partial_{(\alpha\beta}\rho_{\gamma)} = \mathbb{D}^2\rho_\alpha = \bar{\mathbb{D}}^2\rho_\alpha = 0 . \quad (10.7)$$

The general solution of the constraints (10.6) is

$$\rho^\alpha(x, \theta, \bar{\theta}) = \epsilon^\alpha - i\lambda\theta^\alpha + i\bar{\lambda}\bar{\theta}^\alpha + \eta_\beta x^{\alpha\beta} + i\eta^\alpha\theta^\beta\bar{\theta}_\beta . \quad (10.8)$$

Here the real spinor  $\epsilon^\alpha$  generates the third  $Q$ -supersymmetry, the complex scalar  $\lambda$  the off-diagonal  $\text{SU}(2)$  transformation and the real spinor  $\eta^\alpha$  the third  $S$ -supersymmetry transformation. Associated with  $\rho^\alpha$  is a complex scalar  $\boldsymbol{\rho}$  and its complex conjugate  $\bar{\boldsymbol{\rho}}$  defined as follows:

$$\rho_\alpha = \mathbb{D}_\alpha\boldsymbol{\rho} , \quad \rho_\alpha = \bar{\mathbb{D}}_\alpha\bar{\boldsymbol{\rho}} . \quad (10.9)$$

The scalar  $\bar{\boldsymbol{\rho}}$  is defined modulo arbitrary shifts of the form:

$$\bar{\boldsymbol{\rho}} \rightarrow \bar{\boldsymbol{\rho}} + \varphi , \quad \bar{\mathbb{D}}_\alpha\varphi = 0 . \quad (10.10)$$

This freedom is not strong enough to make  $\boldsymbol{\rho}$  and  $\bar{\boldsymbol{\rho}}$  coincide,<sup>14</sup> although  $\boldsymbol{\rho}$  and  $\bar{\boldsymbol{\rho}}$  obey the reality condition:

$$\mathbb{D}_\alpha\boldsymbol{\rho} = \bar{\mathbb{D}}_\alpha\bar{\boldsymbol{\rho}} . \quad (10.11)$$

We have seen that the sigma-model (10.4) is invariant under the  $\mathcal{N} = 2$  superconformal transformations (10.5). Let us now define the extended superconformal transformation:

$$\delta\phi^a = \frac{1}{2}\bar{\mathbb{D}}^2\left\{\bar{\boldsymbol{\rho}}\omega^{ab}\chi_b\right\} . \quad (10.12)$$

Our goal is to prove that the action (10.4) is invariant under (10.12). The variation of the action is

$$\begin{aligned} \delta S &= -\frac{1}{2}\int d^3x d^2\theta d^2\bar{\theta} (\bar{\mathbb{D}}_\alpha\chi_a)(\bar{\mathbb{D}}^\alpha\bar{\boldsymbol{\rho}})\omega^{ab}\chi_b + \text{c.c.} \\ &= -\frac{1}{2}\int d^3x d^2\theta d^2\bar{\theta} \bar{\rho}_\alpha(\bar{\mathbb{D}}^\alpha\bar{\phi}^{\bar{c}})g_{\bar{c}a}\omega^{ab}\chi_b + \text{c.c.} \\ &= -\frac{1}{2}\int d^3x d^2\theta d^2\bar{\theta} \bar{\rho}_\alpha(\bar{\mathbb{D}}^\alpha\bar{\phi}^{\bar{a}})\bar{\omega}_{\bar{a}\bar{b}}\bar{\chi}^{\bar{b}} + \text{c.c.} \end{aligned} \quad (10.13)$$

---

<sup>14</sup>A simple way to justify this statement is as follows. According to the analysis given in appendix C.1,  $\Lambda^{11} = \frac{i}{2}\mathbb{D}_\alpha\rho^\alpha$  and its complex conjugate is  $(\Lambda^{11})^* = \Lambda^{22} = \frac{i}{2}\bar{\mathbb{D}}_\alpha\rho^\alpha$ . These relations can be rewritten in terms of  $\boldsymbol{\rho}$  and  $\bar{\boldsymbol{\rho}}$  as  $\Lambda^{11} = \frac{i}{2}\mathbb{D}_\alpha\bar{\mathbb{D}}^\alpha\bar{\boldsymbol{\rho}}$  and  $\Lambda^{22} = \frac{i}{2}\bar{\mathbb{D}}_\alpha\mathbb{D}^\alpha\boldsymbol{\rho}$ , and thus  $\boldsymbol{\rho}$  has to be complex, for  $\Lambda^{11}$  is complex in general.

Since the tensor fields  $\bar{\omega}_{\bar{a}\bar{b}}$  and  $\bar{\chi}^{\bar{b}}$  are anti-holomorphic, the combination  $\bar{\omega}_{\bar{a}\bar{b}} \bar{\chi}^{\bar{b}}$  appearing in the integrand is antichiral. As a result, doing the Grassmann integral  $\int d^2\theta$  gives

$$\delta S = \frac{1}{4} \int d^3x d^2\bar{\theta} (\mathbb{D}^\alpha \bar{\rho}_\alpha) \bar{\omega}_{\bar{a}\bar{b}} \bar{\chi}^{\bar{b}} \mathbb{D}^\beta \bar{\mathbb{D}}_\beta \bar{\phi}^{\bar{a}} + \frac{1}{8} \int d^3x d^2\bar{\theta} \bar{\rho}_\alpha \bar{\omega}_{\bar{a}\bar{b}} \bar{\chi}^{\bar{b}} \mathbb{D}^2 \bar{\mathbb{D}}^\alpha \bar{\phi}^{\bar{a}} + \text{c.c.} \quad (10.14)$$

In deriving the first term, we have used the identity  $\mathbb{D}_{(\alpha} \rho_{\beta)} = 0$ . The above variation vanishes, for  $\mathbb{D}^\beta \bar{\mathbb{D}}_\beta \bar{\phi} \equiv 0$  and  $\mathbb{D}^2 \bar{\mathbb{D}}_\beta \bar{\phi} \equiv 0$  for any antichiral superfield  $\bar{\phi}$ .

## 10.2 $\mathcal{N} = 4$ superconformal invariance

As shown in Appendix C, subsection C.2, an arbitrary  $\mathcal{N} = 4$  superconformal transformation decomposes into three transformations in  $\mathcal{N} = 2$  superspace: (i) an  $\mathcal{N} = 2$  superconformal transformation generated by an  $\mathcal{N} = 2$  superconformal Killing vector  $\xi$ ; (ii) an extended superconformal transformation generated by a *complex* spinor superfield  $\rho^\alpha(x, \theta, \bar{\theta})$  obeying the constraints (C.11); (iii) a shadow  $U(1)$  rotation. In regard to transformation (ii), the only difference from the  $\mathcal{N} = 3$  case considered earlier is that the parameter  $\rho^\alpha$  is now complex. Since  $\rho^\alpha$  is complex, the general solution of the constraints (C.11) is

$$\rho^\alpha = \epsilon^\alpha + \lambda_R \theta^\alpha - \bar{\lambda}_L \bar{\theta}^\alpha + \bar{\eta}_\beta x^{\alpha\beta} + i\bar{\eta}^\alpha \theta^\beta \bar{\theta}_\beta . \quad (10.15)$$

Here the complex parameter  $\epsilon^\alpha$  generates the third and fourth  $Q$ -supersymmetries; the complex parameters  $\lambda_L$  and  $\lambda_R$  generate off-diagonal  $SU(2)_L$  and  $SU(2)_R$  transformations associated with  $\Lambda_L^{11}$  and  $\Lambda_R^{11}$ ; finally, the complex parameter  $\eta^\alpha$  generates the third and fourth  $S$ -supersymmetry transformations.

The constraints (C.11) imply that we can represent

$$\rho_\alpha = \bar{\mathbb{D}}_\alpha \bar{\rho}_L , \quad \bar{\rho}_\alpha = \mathbb{D}_\alpha \rho_L ; \quad (10.16a)$$

$$\rho_\alpha = \mathbb{D}_\alpha \rho_R , \quad \bar{\rho}_\alpha = \bar{\mathbb{D}}_\alpha \bar{\rho}_R , \quad (10.16b)$$

for some complex scalars  $\rho_L$  and  $\rho_R$ . Unlike the  $\mathcal{N} = 3$  case, the mutually conjugate parameters  $\rho_{L,R}$  and  $\bar{\rho}_{L,R}$  do not obey any additional reality condition like (10.11).

In the  $\mathcal{N} = 4$  case, it is possible to define two types of extended superconformal transformations:

$$\delta_L \phi^a = \frac{1}{2} \bar{\mathbb{D}}^2 \left\{ \bar{\rho}_L \omega^{ab} \chi_b \right\} , \quad (10.17a)$$

$$\delta_R \phi^a = \frac{1}{2} \bar{\mathbb{D}}^2 \left\{ \bar{\rho}_R \omega^{ab} \chi_b \right\} . \quad (10.17b)$$



Such transformations leave the action (10.4) invariant, for the proof given at the end of the previous subsection carries over without any change.

Finally, we define the infinitesimal shadow  $U(1)$  transformation of  $\phi^a$ :

$$\delta\phi^a = -\frac{i}{2}\alpha\chi^a(\phi), \quad \bar{\alpha} = \alpha. \quad (10.18)$$

Because of the identity (10.3), this transformation leaves the action invariant. It should be remarked that the shadow chiral rotation is generated by the Killing vector

$$v = i\chi^a(\phi)\frac{\partial}{\partial\phi^a} - i\bar{\chi}^{\bar{a}}(\bar{\phi})\frac{\partial}{\partial\bar{\phi}^{\bar{a}}}. \quad (10.19)$$

## 11 Conclusion

In this paper we have elaborated on various aspect of three-dimensional  $\mathcal{N} \leq 4$  superconformal sigma-models from the superspace point of view. The original motivation for the research presented in this paper was the desire to explore the additional opportunities offered by superspace as compared with the component analysis given in [4].

We have not studied gauged superconformal sigma-models. The procedures of gauging the target-space isometries for sigma-models formulated in superspace are well-elaborated, see in particular [33, 67], and can be naturally used in three dimensions.

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## A 3D notation and conventions

Our spinor conventions in three space-time dimensions (3D) are compatible with the 4D two-component spinor formalism used by Wess and Bagger [69] and also adopted in

[55]. Specifically, we start from the 4D sigma-matrices

$$(\sigma_{\underline{m}})_{\alpha\dot{\beta}} := (\mathbb{1}, \vec{\sigma}) , \quad \underline{m} = 0, 1, 2, 3 \quad (\text{A.1a})$$

$$(\tilde{\sigma}_{\underline{m}})^{\dot{\alpha}\beta} := (\mathbb{1}, -\vec{\sigma}) , \quad \underline{m} = 0, 1, 2, 3 \quad (\text{A.1b})$$

and delete the matrices with space index  $\underline{m} = 2$ . This leads to the 3D gamma-matrices

$$(\sigma_{\underline{m}})_{\alpha\dot{\beta}} \longrightarrow (\gamma_m)_{\alpha\beta} = (\gamma_m)_{\beta\alpha} = (\mathbb{1}, \sigma_1, \sigma_3) , \quad (\text{A.2a})$$

$$(\tilde{\sigma}_{\underline{m}})^{\dot{\alpha}\beta} \longrightarrow (\gamma_m)^{\alpha\beta} = (\gamma_m)^{\beta\alpha} = \varepsilon^{\alpha\gamma} \varepsilon^{\beta\delta} (\gamma_m)_{\gamma\delta} , \quad (\text{A.2b})$$

where the spinor indices are raised and lowered using the  $\text{SL}(2, \mathbb{R})$  invariant tensors

$$\varepsilon_{\alpha\beta} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} , \quad \varepsilon^{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} , \quad \varepsilon^{\alpha\gamma} \varepsilon_{\gamma\beta} = \delta_{\beta}^{\alpha} \quad (\text{A.3})$$

using the standard rule:

$$\psi^{\alpha} = \varepsilon^{\alpha\beta} \psi_{\beta} , \quad \psi_{\alpha} = \varepsilon_{\alpha\beta} \psi^{\beta} . \quad (\text{A.4})$$

By construction, the matrices  $(\gamma_m)_{\alpha\beta}$  and  $(\gamma_m)^{\alpha\beta}$  are real and symmetric. Using the properties of the 4D sigma-matrices, we can immediately read off the properties of the 3D gamma-matrices. In particular, for the matrices

$$\gamma_m := (\gamma_m)_{\alpha}^{\beta} = \varepsilon^{\beta\gamma} (\gamma_m)_{\alpha\gamma} \quad (\text{A.5})$$

one readily obtains the relations

$$\{\gamma_m, \gamma_n\} = 2\eta_{mn} \mathbb{1} , \quad (\text{A.6a})$$

$$\gamma_m \gamma_n = \eta_{mn} \mathbb{1} + \varepsilon_{mnp} \gamma^p , \quad (\text{A.6b})$$

where the 3D Minkowski metric is  $\eta_{mn} = \eta^{mn} = \text{diag}(-1, 1, 1)$  and the Levi-Civita tensor is normalised as  $\varepsilon_{012} = -\varepsilon^{012} = -1$ . Another useful relation is the following

$$(\gamma^m)_{\alpha\beta} (\gamma_m)_{\gamma\delta} = 2\varepsilon_{\alpha(\gamma} \varepsilon_{\delta)\beta} . \quad (\text{A.7})$$

To comply with the tradition, we will label the 3D vector indices by values 0, 1, 2. Given a three-vector  $V_m$ , it can be equivalently described by a symmetric bi-spinor  $V_{\alpha\beta}$  defined as

$$V_{\alpha\beta} := (\gamma^m)_{\alpha\beta} V_m = V_{\beta\alpha} , \quad V_m = -\frac{1}{2} (\gamma_m)^{\alpha\beta} V_{\alpha\beta} . \quad (\text{A.8})$$

## B The super-Poincaré group

The  $\mathcal{N}$ -extended super-Poincaré group in three space-time dimensions,  $\mathfrak{P}(3|\mathcal{N})$ , can naturally be realised as a subgroup of the superconformal group  $\text{OSp}(\mathcal{N}|2, \mathbb{R})$ . Any element  $g \in \mathfrak{P}(3|\mathcal{N})$  can uniquely be represented in the form:

$$g = s(a, \epsilon) h(M) , \quad (\text{B.1a})$$

$$s(a, \epsilon) = \exp \left( \begin{array}{c|c|c} 0 & 0 & 0 \\ \hline -a^{\alpha\beta} & 0 & -\sqrt{2}\epsilon^\alpha{}_J \\ \hline i\sqrt{2}\epsilon_I{}^\beta & 0 & 0 \end{array} \right) \\ = \left( \begin{array}{c|c|c} \delta_\alpha{}^\beta & 0 & 0 \\ \hline -a^{\alpha\beta} + \frac{i}{2}\epsilon^{\alpha\beta}\epsilon^2 & \delta^\alpha{}_\beta & -\sqrt{2}\epsilon^\alpha{}_J \\ \hline i\sqrt{2}\epsilon_I{}^\beta & 0 & \delta_{IJ} \end{array} \right) , \quad (\text{B.1b})$$

$$h(M) = \exp \left( \begin{array}{c|c|c} M & 0 & 0 \\ \hline 0 & (M^{-1})^T & 0 \\ \hline 0 & 0 & \mathbb{1}_{\mathcal{N}} \end{array} \right) , \quad M \in \text{SL}(2, \mathbb{R}) . \quad (\text{B.1c})$$

In eq. (B.1b), the bosonic  $a^{\alpha\beta} = a^{\beta\alpha} = a^m(\gamma_m)^{\alpha\beta}$  and fermionic  $\epsilon_I{}^\alpha = \epsilon^\alpha{}_I \equiv \epsilon_I^\alpha$  parameters are real.

$\mathcal{N}$ -extended Minkowski superspace is the homogeneous space

$$\mathbb{M}^{3|2\mathcal{N}} = \mathfrak{P}(3|\mathcal{N})/\text{SL}(2, \mathbb{R}) , \quad (\text{B.2})$$

where  $\text{SL}(2, \mathbb{R})$  is identified with the set of all matrices  $h(M)$  defined in (B.1c). The points of  $\mathbb{M}^{3|2\mathcal{N}}$  can be parametrised by the variables

$$z^M = (x^m, \theta_I^\alpha) \quad (\text{B.3})$$

which correspond to the following coset representative:

$$s(z) := s(x, \theta) = \left( \begin{array}{c|c|c} \delta_\alpha{}^\beta & 0 & 0 \\ \hline -x^{\alpha\beta} + \frac{i}{2}\epsilon^{\alpha\beta}\theta^2 & \delta^\alpha{}_\beta & -\sqrt{2}\theta^\alpha{}_J \\ \hline i\sqrt{2}\theta_I{}^\beta & 0 & \delta_{IJ} \end{array} \right) , \quad x^{\alpha\beta} = x^m(\gamma_m)^{\alpha\beta} . \quad (\text{B.4})$$

The supersymmetry transformation  $s(0, \epsilon)$  acts on the superspace according to the law  $s(x, \theta) \rightarrow s(x', \theta') = s(0, \epsilon)s(x, \theta)$ , and thus

$$x'^{\alpha\beta} = x^{\alpha\beta} + i(\epsilon_I^\alpha \theta_I^\beta + \epsilon_I^\beta \theta_I^\alpha) , \quad \theta_I'^\alpha = \theta_I^\alpha + \epsilon_I^\alpha . \quad (\text{B.5})$$

These results can be rewritten in terms of  $z^A = (x^a, \theta_I^\alpha)$  as

$$z'^A = z^A - i \epsilon_J^\beta Q_\beta^J z^A , \quad (\text{B.6})$$

where we have introduced the supersymmetry generators

$$Q_\alpha^I = i \frac{\partial}{\partial \theta_I^\alpha} + (\gamma^m)_{\alpha\beta} \theta_I^\beta \partial_m = i \frac{\partial}{\partial \theta_I^\alpha} + \theta_I^\beta \partial_{\beta\alpha} . \quad (\text{B.7})$$

From here we immediately read off the spinor covariant derivatives

$$D_\alpha^I = \frac{\partial}{\partial \theta_I^\alpha} + i(\gamma^m)_{\alpha\beta} \theta_I^\beta \partial_m = \frac{\partial}{\partial \theta_I^\alpha} + i \theta_I^\beta \partial_{\beta\alpha} , \quad (\text{B.8})$$

which obey the anti-commutation relations

$$\{D_\alpha^I, D_\beta^J\} = 2i \delta^{IJ} (\gamma^m)_{\alpha\beta} \partial_m . \quad (\text{B.9})$$

As compared with the supersymmetry in four dimensions, the spinor covariant derivatives possess unusual conjugation properties. Specifically, given an arbitrary superfield  $F$  and  $\bar{F} := (F)^*$  its complex conjugate, the following relations holds

$$(D_\alpha^I F)^* = -(-1)^{\epsilon(F)} D_\alpha^I \bar{F} , \quad (\text{B.10})$$

where  $\epsilon(F)$  denotes the Grassmann parity of  $F$ .

## C $\mathcal{N} = 2$ reduction for $\mathcal{N} = 3$ and $\mathcal{N} = 4$ superconformal Killing vector fields

The  $\mathcal{N}$ -extended superconformal Killing vector fields were studied in section 5. Here we describe the reduction of the  $\mathcal{N} = 3$  and  $\mathcal{N} = 4$  superconformal Killing vectors to  $\mathcal{N} = 2$  superspace. The results of this appendix are used in the sections 7, 9 and 10. The analysis and the results given below are analogous to those described in the the 4D  $\mathcal{N} = 2$  case in [12, 14].

We recall that  $\mathcal{N} = 3$  superspace is parametrised by the coordinates  $z^A = (x^a, \theta_{ij}^\alpha)$ , while its  $\mathcal{N} = 4$  cousin by  $z^A = (x^a, \theta_{ij}^\alpha)$ . Embedded into these superspaces is the  $\mathcal{N} = 2$  superspace parametrised by variables  $z^{\mathfrak{A}} = (x^a, \theta^\alpha, \bar{\theta}^\alpha)$ , where the odd complex coordinate  $\theta^\alpha$  is defined as  $\theta^\alpha = \theta_{11}^\alpha = \theta_{1\bar{1}}^\alpha$ , and for its conjugate  $\bar{\theta}^\alpha = (\theta^\alpha)^*$  we have  $\bar{\theta}^\alpha = \theta_{22}^\alpha = \theta_{2\bar{2}}^\alpha$ .

Let  $\Phi(z^A)$  be an arbitrary  $\mathcal{N} = 3$  or  $\mathcal{N} = 4$  superfield. Its  $\mathcal{N} = 2$  projection is

$$\Phi| := \Phi(z^A)|_{\theta_\perp=0} , \quad (\text{C.1})$$

where  $\theta_\perp$  stands for  $\theta_{12}^\alpha$  in the  $\mathcal{N} = 3$  case, and  $(\theta_{12}^\alpha, \theta_{21}^\alpha)$  in the  $\mathcal{N} = 4$  case. Given a vector field  $V = V^A(z)D_A$  on the  $\mathcal{N} = 3$  or  $\mathcal{N} = 4$  superspace, its  $\mathcal{N} = 2$  projection is defined as

$$V| = V^A| D_A . \quad (\text{C.2})$$

The important point is that the covariant derivatives  $D_\alpha^{11} = D_\alpha^{1\bar{1}}$  and  $D_\alpha^{22} = D_\alpha^{2\bar{2}}$  depend only on  $\theta^\alpha$  and  $\bar{\theta}^\alpha$ . The explicit representation of the  $\mathcal{N} = 2$  covariant derivatives is

$$\mathbb{D}_\alpha = \frac{\partial}{\partial \theta^\alpha} + i\bar{\theta}^\beta \partial_{\alpha\beta} = D_\alpha^{11} = D_\alpha^{1\bar{1}} , \quad (\text{C.3a})$$

$$\bar{\mathbb{D}}_\alpha = -\frac{\partial}{\partial \bar{\theta}^\alpha} - i\theta^\beta \partial_{\alpha\beta} = -D_\alpha^{22} = -D_\alpha^{2\bar{2}} . \quad (\text{C.3b})$$

They satisfy the anticommutation relations (5.18).

## C.1 $\mathcal{N} = 3$ superconformal Killing vector fields

Let  $\xi$  be a  $\mathcal{N} = 3$  superconformal Killing vector. Consider its  $\mathcal{N} = 2$  projection

$$\xi| := \xi^A| D_A = \xi + 2i\rho^\alpha D_\alpha^{12} , \quad \xi = \xi^a \partial_a + \xi^\alpha \mathbb{D}_\alpha - \bar{\xi}^\alpha \bar{\mathbb{D}}_\alpha . \quad (\text{C.4})$$

Making use of eq. (5.34) gives

$$[\xi, D_\alpha^{11}]| = [\xi|, \mathbb{D}_\alpha] = \omega_\alpha{}^\beta \mathbb{D}_\beta + \frac{1}{4}(\sigma - 3\bar{\sigma})\mathbb{D}_\alpha - \Lambda^{11}| D_\alpha^{12} . \quad (\text{C.5})$$

The parameters in (C.4) and (C.5) are related to those in (5.34) as follows:

$$\xi^a := \xi^a| , \quad \xi^\alpha := \xi_{11}^\alpha| , \quad \bar{\xi}^\alpha := \xi_{22}^\alpha| , \quad \rho^\alpha := -i\xi_{12}^\alpha| = \bar{\rho}^\alpha , \quad (\text{C.6a})$$

$$\sigma| = \frac{1}{3}\partial^a \xi_a = \frac{1}{2}(\mathbb{D}_\alpha \xi^\alpha - \bar{\mathbb{D}}_\alpha \bar{\xi}^\alpha) , \quad \omega_{\alpha\beta}| = \omega_{\alpha\beta} = -\mathbb{D}_{(\alpha} \xi_{\beta)} = \bar{\mathbb{D}}_{(\alpha} \bar{\xi}_{\beta)} , \quad (\text{C.6b})$$

$$\Lambda^{11}| = \frac{i}{2}\mathbb{D}_\alpha \rho^\alpha , \quad \Lambda^{12}| = -\frac{1}{8}(\mathbb{D}_\alpha \xi^\alpha + \bar{\mathbb{D}}_\alpha \bar{\xi}^\alpha) , \quad (\text{C.6c})$$

$$\sigma = \frac{1}{4}(\mathbb{D}_\alpha \xi^\alpha - 3\bar{\mathbb{D}}_\alpha \bar{\xi}^\alpha) = (2\Lambda^{12}| + \sigma|) , \quad \bar{\mathbb{D}}_\alpha \sigma = 0 . \quad (\text{C.6d})$$

The  $\mathcal{N} = 3$  superconformal transformation generated by  $\xi$  induces two different transformations of  $\mathcal{N} = 2$  superfields. They are:

1. A  $\mathcal{N} = 2$  superconformal transformation. It is generated by  $\xi|$  which, due to (C.5),

is a  $\mathcal{N} = 2$  superconformal vector field. Its components  $\xi^a, \xi^\alpha, \bar{\xi}^\alpha$  as well as the descendants  $\omega_{\alpha\beta}, \sigma|$  and  $\Lambda^{12}|$ , which are introduced above, correspond to the  $\mathcal{N} = 2$  parameters  $\xi^a, \xi^\alpha, \bar{\xi}^\alpha, \omega_{\alpha\beta}, \sigma$  and  $(i/2)\Lambda$  of subsection (5.3).

**2.** An extended superconformal transformation. It is generated by the real spinor parameter  $\rho^\alpha$  under the constraints

$$\mathbb{D}_{(\alpha}\rho_{\beta)} = \bar{\mathbb{D}}_{(\alpha}\rho_{\beta)} = 0 \implies \partial_{(\alpha\beta}\rho_{\gamma)} = \mathbb{D}^2\rho_\alpha = \bar{\mathbb{D}}^2\rho_\alpha = 0 . \quad (\text{C.7})$$

The general solution to these constraints is

$$\rho^\alpha = \epsilon^\alpha - i\lambda\theta^\alpha + i\bar{\lambda}\bar{\theta}^\alpha + \eta_\beta x^{\alpha\beta} + i\eta^\alpha\theta^\beta\bar{\theta}_\beta , \quad (\text{C.8})$$

with  $\epsilon^\alpha, \lambda$  and  $\eta^\alpha$  constant parameters. The *real* parameter  $\epsilon^\alpha$  generates the third  $Q$ -supersymmetry; the *complex* parameter  $\lambda = \Lambda^{11}|_{\theta=0}$  generates an off-diagonal  $\text{SU}(2)$  transformation; finally, the *real* parameter  $\eta^\alpha$  generates the third  $S$ -supersymmetry transformation.

## C.2 $\mathcal{N} = 4$ superconformal Killing vector fields

In the  $\mathcal{N} = 4$  case, the analysis is similar to that given in the previous subsection. The  $\mathcal{N} = 2$  projection of the  $\mathcal{N} = 4$  superconformal Killing vector field is

$$\xi| := \xi^A|D_A = \xi + \rho^\alpha D_\alpha^{1\bar{2}} - \bar{\rho}^\alpha D_\alpha^{2\bar{1}} , \quad \xi = \xi^a\partial_a + \xi^\alpha\mathbb{D}_\alpha - \bar{\xi}^\alpha\bar{\mathbb{D}}_\alpha \quad (\text{C.9a})$$

$$[\xi|, \mathbb{D}_\alpha] = \omega_\alpha{}^\beta\mathbb{D}_\beta + \frac{1}{4}(\sigma - 3\bar{\sigma})\mathbb{D}_\alpha - (\Lambda_L)^{11}|D_\alpha^{2\bar{1}} - (\Lambda_R)^{\bar{1}\bar{1}}|D_\alpha^{1\bar{2}} , \quad (\text{C.9b})$$

where

$$\xi^a := \xi^a| , \quad \xi^\alpha := \xi_{1\bar{1}}^\alpha| , \quad \rho^\alpha := \xi_{1\bar{2}}^\alpha| , \quad (\text{C.10a})$$

$$\sigma| = \frac{1}{3}\partial^a\xi_a = \frac{1}{2}(\mathbb{D}_\alpha\xi^\alpha - \bar{\mathbb{D}}_\alpha\bar{\xi}^\alpha) , \quad \omega_{\alpha\beta} = \omega_{\alpha\beta}| = -\mathbb{D}_{(\alpha}\xi_{\beta)} = \bar{\mathbb{D}}_{(\alpha}\bar{\xi}_{\beta)} , \quad (\text{C.10b})$$

$$\Lambda_L^{11}| = -\frac{1}{2}\mathbb{D}_\alpha\bar{\rho}^\alpha , \quad \Lambda_R^{\bar{1}\bar{1}}| = \frac{1}{2}\mathbb{D}_\alpha\rho^\alpha , \quad (\text{C.10c})$$

$$\Lambda_L^{12}| + \Lambda_R^{1\bar{2}}| = -\frac{1}{4}(\mathbb{D}_\alpha\xi^\alpha + \bar{\mathbb{D}}_\alpha\bar{\xi}^\alpha) , \quad (\text{C.10d})$$

$$\Lambda_L^{12}| - \Lambda_R^{1\bar{2}}| = \frac{1}{4}(D_\alpha^{2\bar{1}}\xi_{2\bar{1}}^\alpha| - D_\alpha^{1\bar{2}}\xi_{1\bar{2}}^\alpha|) , \quad (\text{C.10e})$$

$$\sigma = \frac{1}{4}(\mathbb{D}_\alpha\xi^\alpha - 3\bar{\mathbb{D}}_\alpha\bar{\xi}^\alpha) = \Lambda_L^{12}| + \Lambda_R^{1\bar{2}}| + \sigma| , \quad \bar{\mathbb{D}}_\alpha\sigma = 0 . \quad (\text{C.10f})$$

Associated with  $\xi$  are three types of  $\mathcal{N} = 2$  superfields, specifically:

**1.** A  $\mathcal{N} = 2$  superconformal transformation. It is generated by the  $\mathcal{N} = 2$  superconformal vector field  $\xi$ . Its components  $\xi^a, \xi^\alpha, \bar{\xi}^\alpha$  and the descendants  $\omega_{\alpha\beta}, \sigma|$  and

$(\Lambda_L^{12} + \Lambda_R^{12})$  should be respectively identified with the  $\mathcal{N} = 2$  superconformal parameters  $\xi^a, \xi^\alpha, \bar{\xi}^\alpha, \omega_{\alpha\beta}, \sigma$  and  $(i/2)\Lambda$  introduced in subsection (5.3).

**2.** An extended superconformal transformation. It is generated by the complex parameter  $\rho^\alpha$  satisfying the constraints

$$\mathbb{D}_{(\alpha}\rho_{\beta)} = \bar{\mathbb{D}}_{(\alpha}\rho_{\beta)} = 0 \quad \implies \quad \partial_{(\alpha\beta}\rho_{\gamma)} = \mathbb{D}^2\rho_\alpha = \bar{\mathbb{D}}^2\rho_\alpha = 0 . \quad (\text{C.11})$$

The constraints are solved by

$$\rho^\alpha = \epsilon^\alpha + \lambda_R \theta^\alpha - \bar{\lambda}_L \bar{\theta}^\alpha + \bar{\eta}_\beta x^{\alpha\beta} + i\bar{\eta}^\alpha \theta^\beta \bar{\theta}_\beta . \quad (\text{C.12})$$

Here the complex parameter  $\epsilon^\alpha$  generates the third and fourth  $Q$ -supersymmetries; the complex parameters  $\lambda_L = \Lambda_L^{11}|_{\theta=0}$  and  $\lambda_R = \Lambda_R^{11}|_{\theta=0}$  generate off-diagonal  $\text{SU}(2)_L$  and  $\text{SU}(2)_R$  transformations; finally, the complex parameter  $\eta^\alpha$  generates the third and fourth  $S$ -supersymmetry transformations.

**3.** A shadow  $\text{U}(1)$  rotation is generated by

$$\alpha := i(\Lambda_R^{12} - \Lambda_L^{12}) = \text{const} . \quad (\text{C.13})$$

In  $\mathcal{N} = 4$  superspace, it describes a  $\text{U}(1)$  phase transformation of  $\theta_{12}^\alpha, \theta_{21}^\alpha$  only, with  $\theta_{11}^\alpha, \theta_{22}^\alpha$  kept unchanged.

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